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Chapter 7

Applications of Differential Calculus

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"Nothing takes place in the world whose meaning is not that of some maximum or minimum"

- Leonhard Euler

7.1 Introduction

7.1.1 Early Developments

Rudolf Otto Sigismund Lipschitz
1832-1903

The primary objective of differential calculus is to partition something into smaller parts (infinitesimal parts), in order to determine how it changes. For this reason today's differential calculus was earlier named as **infinitesimal calculus**. Applications of differential calculus to problems in physics and astronomy was contemporary with the origin of science. All through the 18^{th} century these applications were multiplied until Laplace and Lagrange, towards the end of the 18th century, had brought the whole range of the study of forces into the realm of analysis.

The development of applications of differentiation are also due to Lejeune Dirichlet, Riemann, von Neumann, Heine, Kronecker, Lipschitz, Christoffel, Kirchhoff, Beltrami, and many of the leading physicists of the century.

- Differential calculus has applications in geometry and dynamics.
- Derivatives of function, representing cost, strength, materials in a process, profit, etc., are used to determine the monotonicity of functions and there by to determine the extreme values of the quantities represented by those functions.
- Derivatives of a function do find a prominent place in many of the modelling problems in engineering and sciences.
- Differential calculus has applications in social sciences and medical sciences too.

Using just the first two derivatives of a function $f(x)$, in this chapter, the nature of the function, sketching of curve $y = f(x)$, and local extrema (maxima or minima) of $f(x)$ are determined. Further, using certain higher derivatives of $f(x)$ (if they exist), series expansion of $f(x)$ about a point are also discussed.

Learning Objectives

Upon completion of this chapter, students will be able to

- apply derivatives to geometrical problems
- use derivatives to physical problems
- identify the nature of curves like monotonicity, convexity, and concavity
- model real time problems for computing the extreme values using derivatives
- trace the curves for polynomials and other functions.

7.2 Meaning of Derivatives

7.2.1 Derivative as slope

Slope or Gradient of a line: Let l be any given non vertical line as in the Fig. 7.1. Taking a finite horizontal line segment of any length with the starting point in the given line l and the vertical line segment starting from the end of the horizontal line to touch the given line. It can be observed that the ratio of the vertical length to the horizontal length is always a constant. This ratio is called the slope of the line l and it is denoted as m.

The slope can be used as a measure to determine the increasing or decreasing nature of a line. The line is said to be increasing or decreasing according as $m > 0$ or $m < 0$ respectively. When $m = 0$, the value of y does not change. Recall that $y = mx + c$ represents a straight line in the XY plane where m denotes the slope of the line.

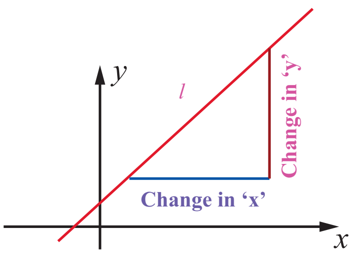


Fig.7.1

Slope or Gradient of a curve: Let $y = f(x)$ be a given curve. The slope of the line joining the two distinct points $(x, f(x))$ and the point $(x+h, f(x+h))$ is

$$\frac{f(x+h)-f(x)}{h}. \text{ (Newton quotient).} \qquad ...(1)$$

Taking the limit as $h \to 0$, we get

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = f'(x), \text{ (limit of Newton quotient)} \qquad ... (2)$$

which is the slope of the curve at the point (x, y) or $(x, f(x))$.

Remark

If θ is the angle made by the tangent to the curve $y = f(x)$ at the point (x, y), then the slope of the curve at (x, y) is $f'(x) = \tan\theta$, where θ is measured in the anti-clockwise direction from the X-axis. Note that, $f'(x)$ is also denoted by $\frac{dy}{dx}$ and also called *instantaneous rate of change*. The average rate of change in an interval is calculated using Newton quotient.

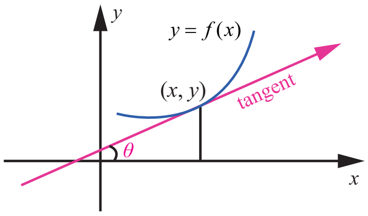


Fig. 7.2

Example 7.1

For the function $f(x) = x^2, x \in [0,2]$ compute the average rate of changes in the subintervals $[0,0.5],[0.5,1],[1,1.5],[1.5,2]$ and the instantaneous rate of changes at the points $x = 0.5, 1, 1.5, 2$.

Solution

The average rate of change in an interval $[a,b]$ is $\frac{f(b)-f(a)}{b-a}$ whereas, the instantaneous rate of change at a point x is $f'(x)$ for the given function. They are respectively, $b+a$ and $2x$.

Rate of changes

a	b	x	Average rate is $\dfrac{f(b)-f(a)}{b-a} = b+a$	Instantaneous rate is $f'(x) = 2x$
0	0.5	0.5	**0.5**	**1**
0.5	1	1	**1.5**	**2**
1	1.5	1.5	**2.5**	**3**
1.5	2	2	**3.5**	**4**

Table 7.1

7.2.2 Derivative as rate of change

We have seen how the derivative is used to determine slope. The derivative can also be used to determine the rate of change of one variable with respect to another. A few examples are population growth rates, production rates, water flow rates, velocity, and acceleration.

A common use of rate of change is to describe the motion of an object moving in a straight line. In such problems, it is customary to use either a horizontal or a vertical line with a designated origin to represent the line of motion. On such lines, movements in the forward direction considered to be in the positive direction and movements in the backward direction is considered to be in the negative direction.

The function $s(t)$ that gives the position (relative to the origin) of an object as a function of time t is called a position function. It is denoted by $s = f(t)$. The velocity and the acceleration at time t is denoted as $v(t) = \dfrac{ds}{dt}$, and $a(t) = \dfrac{dv}{dt} = \dfrac{d^2s}{dt^2}$.

Remark

The following remarks are easy to observe:

(1) Speed is the absolute value of velocity regardless of direction and hence,

Speed $= |v(t)| = \left|\dfrac{ds}{dt}\right|$.

(2) • When the particle is at rest then $v(t) = 0$.

• When the particle is moving forward then $v(t) > 0$.

• When the particle is moving backward then $v(t) < 0$.

• When the particle changes direction, $v(t)$ then changes its sign.

(3) If t_c is the time point between the time points t_1 and t_2 $(t_1 < t_c < t_2)$ where the particle changes direction then the total distance travelled from time t_1 to time t_2 is calculated as

$|s(t_1) - s(t_c)| + |s(t_c) - s(t_2)|$.

(4) Near the surface of the planet Earth, all bodies fall with the same constant acceleration. When air resistance is absent or insignificant and only force acting on a falling body is the force of gravity, we call the way the body falls is a free fall.

An object thrown at time $t=0$ from initial height s_0 with initial velocity v_0 satisfies the equation.

$$a=-g,\quad v=-gt+v_0,\quad s=-\frac{gt^2}{2}+v_0t+s_0.$$

where, $g=9.8\ \text{m}/\text{s}^2$ or $32\ \text{ft}/\text{s}^2$.

A few examples of **quantities** which are the **rates of change with respect to some other quantity** in our daily life are given below:

1. Slope is the **rate of change in vertical length with respect to horizontal length.**
2. Velocity is the **rate of displacement with respect to time**.
3. Acceleration is the **rate of change in velocity with respect to time**.
4. The steepness of a hillside is the **rate of change in its elevation with respect to linear distance**.

Consider the following two situations:

- A person is continuously driving a car from Chennai to Dharmapuri. The distance (measured in kilometre) travelled is expressed as a function of time (measured in hours) by $D(t)$. What is the meaning one can attribute to $D'(3)=70$?

  It means that, **"the rate of distance when $t=3$ is 70 kmph"**.

- A water source is draining with respect to the time t. The amount of water so drained after t days is expressed as $V(t)$. What is the meaning of the slope of the tangent to the curve $y=V(t)$ at $t=7$ is -3?

  It means that, **"the water is draining at the rate of 3 units per day on day 7"**.

Likewise the rate of change concept can be used in our daily life problems. Let us now illustrate this with more examples.

Example 7.2

The temperature T in celsius in a long rod of length 10 m, insulated at both ends, is a function of length x given by $T=x(10-x)$. Prove that the rate of change of temperature at the midpoint of the rod is zero.

Solution

We are given that, $T=10x-x^2$. Hence, the rate of change at any distance from one end is given by $\frac{dT}{dx}=10-2x$. The mid point of the rod is at $x=5$. Substituting $x=5$, we get $\frac{dT}{dx}=0$. ■

Example 7.3

A person learnt 100 words for an English test. The number of words the person remembers in t days after learning is given by $W(t)=100\times(1-0.1t)^2$, $0\le t\le 10$. What is the rate at which the person forgets the words 2 days after learning?

Solution

We have,

$$\frac{d}{dt}W(t) = -20\times(1-0.1t).$$

Therefore at $t=2$, $\frac{d}{dt}W(t) = -16$.

That is, the person forgets at the rate of 16 words after 2 days of studying. ■

Example 7.4

A particle moves so that the distance moved is according to the law $s(t)=\frac{t^3}{3}-t^2+3$. At what time the velocity and acceleration are zero.

Solution

$$\text{Distance moved in time } 't' \text{ is } s = \frac{t^3}{3}-t^2+3.$$

$$\text{Velocity at time } 't' \text{ is } v(t) = \frac{ds}{dt}=t^2-2t.$$

$$\text{Acceleration at time } 't' \text{ is } a(t) = \frac{dV}{dt}=2t-2.$$

Therefore, the velocity is zero when $t^2-2t=0$, that is $t=0,2$. The acceleration is zero when $2t-2=0$. That is at time $t=1$.

Example 7.5

A particle is fired straight up from the ground to reach a height of s feet in t seconds,where $s(t)=128t-16t^2$.

(i) Compute the maximum height of the particle reached.

(ii) What is the velocity when the particle hits the ground?

Solution

(i) At the maximum height, the velocity $v(t)$ of the particle is zero.

Now, we find the velocity of the particle at time t.

$v(t)=\frac{ds}{dt}=128-32t$

$v(t)=0 \Rightarrow 128-32t=0 \Rightarrow t=4$.

After 4 seconds, the particle reaches the maximum height.

The height at $t=4$ is $s(4)=128(4)-16(4)^2=256$ ft.

(ii) When the particle hits the ground then $s=0$.

$s=0 \Rightarrow 128t-16t^2=0$

$\Rightarrow t=0,\ 8$ seconds.

The particle hits the ground at $t=8$ seconds. The velocity when it hits the ground is $v(8)=-128$ ft/s.

Example 7.6

A particle moves along a horizontal line such that its position at any time $t\geq 0$ is given by $s(t)=t^3-6t^2+9t+1$, where s is measured in metres and t in seconds?

(i) At what time the particle is at rest?

(ii) At what time the particle changes its direction?

(iii) Find the total distance travelled by the particle in the first 2 seconds.

Solution

Given that $s(t)=t^3-6t^2+9t+1$. On differentiating, we get $v(t)=3t^2-12t+9$ and $a(t)=6t-12$.

(i) The particle is at rest when $v(t)=0$. Therefore, $v(t)=3(t-1)(t-3)=0$ gives $t=1$and $t=3$.

(ii) The particle changes its direction when $v(t)$ changes its sign. Now.

if $0 \le t < 1$ then both $(t-1)$ and $(t-3) < 0$ and hence, $v(t) > 0$.

If $1 < t < 3$ then $(t-1) > 0$ and $(t-3) < 0$ and hence, $v(t) < 0$.

If $t > 3$ then both $(t-1)$ and $(t-3) > 0$ and hence, $v(t) > 0$.

Therefore, the particle changes its direction when $t = 1$ and $t = 3$.

(iii) The total distance travelled by the particle from time $t = 0$ to $t = 2$ is given by, $|s(0)-s(1)| + |s(1)-s(2)| = |1-5| + |5-3| = 6$ metres. ■

7.2.3 Related rates

A related rates problem is a problem which involves at least two changing quantities and asks you to figure out the rate at which one is changing given sufficient information on all of the others. For instance, when two vehicles drive in different directions we should be able to deduce the speed at which they are separating if we know their individual speeds and directions.

Example 7.7

If we blow air into a balloon of spherical shape at a rate of $1000\,\text{cm}^3$ per second, at what rate the radius of the baloon changes when the radius is 7cm? Also compute the rate at which the surface area changes.

B6L6J2

Solution

The volume of the baloon of radius r is $V = \frac{4}{3}\pi r^3$.

We are given $\frac{dV}{dt} = 1000$ and we need to find $\frac{dr}{dt}$ when $r = 7$.

$$\text{Now, } \frac{dV}{dt} = 3 \times \frac{4}{3}\pi r^2 \times \frac{dr}{dt}.$$

Substituting $r = 7$ and $\frac{dV}{dt} = 1000$, we get $1000 = 4\pi \times 49 \times \frac{dr}{dt}$.

$$\text{Hence, } \frac{dr}{dt} = \frac{1000}{4 \times 49 \times \pi} = \frac{250}{49\pi}.$$

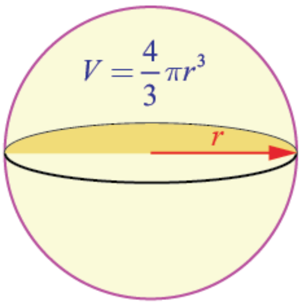


Fig.7.4

The surface area S of the baloon is $S = 4\pi r^2$. Therefore, $\frac{dS}{dt} = 8\pi \times r \times \frac{dr}{dt}$.

Substituting $\frac{dr}{dt} = \frac{250}{49\pi}$ and $r = 7$, we get

$$\frac{dS}{dt} = 8\pi \times 7 \times \frac{250}{49\pi} = \frac{2000}{7}.$$

Therefore, the rate of change of radius is $\frac{250}{49\pi}$ cm/sec and the rate of change of surface area is $\frac{2000}{7}\ \text{cm}^2/\text{sec}$. ■

Example 7.8

The price of a product is related to the number of units available (supply) by the equation $Px+3P-16x=234$, where P is the price of the product per unit in Rupees(₹) and x is the number of units. Find the rate at which the price is changing with respect to time when 90 units are available and the supply is increasing at a rate of 15 units/week.

Solution

We have,

$$P = \frac{234+16x}{x+3}$$

$$\text{Therefore, } \frac{dP}{dt} = -\frac{186}{(x+3)^2}\times\frac{dx}{dt}.$$

Substituting $x=90,\ \frac{dx}{dt}=15$, we get $\frac{dP}{dt}=-\frac{186}{93^2}\times 15=-\frac{10}{31}\approx -0.32$ rupee/week. That is the price is changing, in fact decreasing at the rate of ₹ 0.32 per week. ■

Example 7.9

Salt is poured from a conveyer belt at a rate of 30 cubic metre per minute forming a conical pile with a circular base whose height and diameter of base are always equal. How fast is the height of the pile increasing when the pile is 10 metre high?

Solution

Let h and r be the height and the base radius. Therefore $h=2r$. Let V be the volume of the salt cone.

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{12}\pi h^3;\ \frac{dV}{dt} = 30 \text{ m}^3/\text{min}.$$

$$\text{Hence, } \frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$$

$$\text{Therefore, } \frac{dh}{dt} = 4\frac{dV}{dt}\cdot\frac{1}{\pi h^2}$$

$$\text{That is, } \frac{dh}{dt} = 4\times 30\times\frac{1}{100\pi}$$

$$= \frac{6}{5\pi} \text{ m / min.}$$

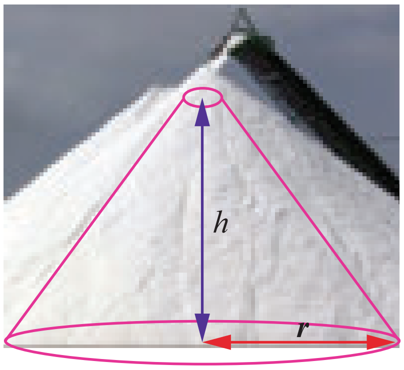


Fig.7.5

■

Example 7.10 (Two variable related rate problem)

A road running north to south crosses a road going east to west at the point P. Car A is driving north along the first road, and car B is driving east along the second road. At a particular time car A is 10 kilometres to the north of P and traveling at 80 km/hr, while car B is 15 kilometres to the east of P and traveling at 100 km/hr. How fast is the distance between the two cars changing?

Solution

Let $a(t)$ be the distance of car A north of P at time t, and $b(t)$ the distance of car B east of P at time t, and let $c(t)$ be the distance from car A to car B at time t. By the Pythagorean Theorem, $c(t)^2 = a(t)^2 + b(t)^2$.

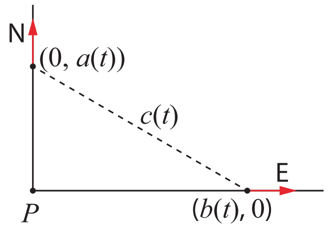


Fig.7.6

Taking derivatives, we get $2c(t)c'(t) = 2a(t)a'(t) + 2b(t)b'(t)$.

$$\text{So, } c' = \frac{aa' + bb'}{c} = \frac{aa' + bb'}{\sqrt{a^2 + b^2}}$$

Substituting known values, we get

$$c' = \frac{(10 \times 80) + (15 \times 100)}{\sqrt{10^2 + 15^2}} = \frac{460}{\sqrt{13}} \approx 127.6 \text{ km/hr}$$

at the time of intersect. ■

EXERCISE 7.1

1. A particle moves along a straight line in such a way that after t seconds its distance from the origin is $s = 2t^2 + 3t$ metres.
   (i) Find the average velocity between $t = 3$ and $t = 6$ seconds.
   (ii) Find the instantaneous velocities at $t = 3$ and $t = 6$ seconds.
2. A camera is accidentally knocked off an edge of a cliff 400 ft high. The camera falls a distance of $s = 16t^2$ in t seconds.
   (i) How long does the camera fall before it hits the ground?
   (ii) What is the average velocity with which the camera falls during the last 2 seconds?
   (iii) What is the instantaneous velocity of the camera when it hits the ground?
3. A particle moves along a line according to the law $s(t) = 2t^3 - 9t^2 + 12t - 4$, where $t \geq 0$.
   (i) At what times the particle changes direction?
   (ii) Find the total distance travelled by the particle in the first 4 seconds.
   (iii) Find the particle's acceleration each time the velocity is zero.
4. If the volume of a cube of side length x is $v = x^3$. Find the rate of change of the volume with respect to x when $x = 5$ units.
5. If the mass $m(x)$ (in kilograms) of a thin rod of length x (in metres) is given by, $m(x) = \sqrt{3x}$ then what is the rate of change of mass with respect to the length when it is $x = 3$ and $x = 27$ metres.
6. A stone is dropped into a pond causing ripples in the form of concentric circles. The radius r of the outer ripple is increasing at a constant rate at 2 cm per second. When the radius is 5 cm find the rate of changing of the total area of the disturbed water?
7. A beacon makes one revolution every 10 seconds. It is located on a ship which is anchored 5 km from a straight shore line. How fast is the beam moving along the shore line when it makes an angle of $45°$ with the shore?
8. A conical water tank with vertex down of 12 metres height has a radius of 5 metres at the top. If water flows into the tank at a rate 10 cubic m/min, how fast is the depth of the water increases when the water is 8 metres deep?
9. A ladder 17 metre long is leaning against the wall. The base of the ladder is pulled away from the wall at a rate of 5 m/s. When the base of the ladder is 8 metres from the wall,
   (i) how fast is the top of the ladder moving down the wall?
   (ii) at what rate, the area of the triangle formed by the ladder, wall, and the floor, is changing?

10, A police jeep, approaching an orthogonal intersection from the northern direction, is chasing a speeding car that has turned and moving straight east. When the jeep is 0.6 km north of the intersection and the car is 0.8 km to the east. The police determine with a radar that the distance between them and the car is increasing at 20 km/hr. If the jeep is moving at 60 km/hr at the instant of measurement, what is the speed of the car?

7.2.4 Equations of Tangent and Normal

According to Leibniz, tangent is the line through a pair of very close points on the curve.

Definition 7.1

The tangent line (or simply tangent) to a plane curve at a given point is the straight line that **just touches** the curve at that point.

Definition 7.2

The normal at a point on the curve is the straight line which is perpendicular to the tangent at that point.

The tangent and the normal of a curve at a point are illustrated in Fig. 7.7.

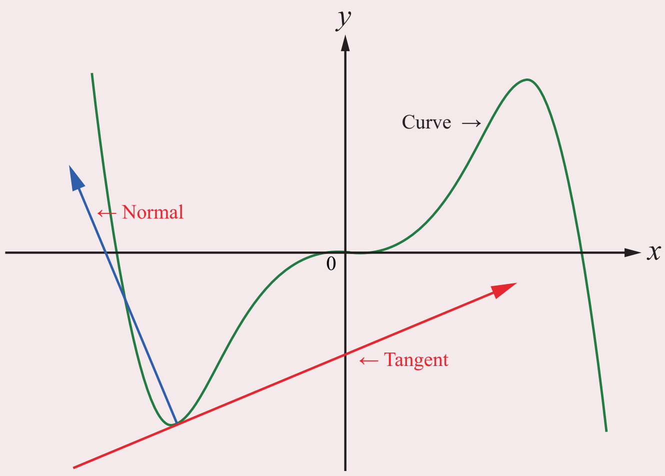


Fig.7.7

Consider the given curve $y = f(x)$.

The equation of the tangent to the curve at the point, say (a,b), is given by

$$y - b = (x-a)\times\left(\frac{dy}{dx}\right)_{(a,b)} \text{ or } y - b = f'(a)\cdot(x-a).$$

In order to get the equation of the normal to the same curve at the same point, we observe that normal is perpendicular to the tangent at the point. Therefore, the slope of the normal at (a,b) is the negative of the reciprocal of the slope of the tangent which is $-\left(\frac{1}{\frac{dy}{dx}}\right)_{(a,b)}$.

Hence, the equation of the normal is ,

$$(y-b) = -\left(\frac{1}{\frac{dy}{dx}}\right)_{(a,b)} \times (x-a) \text{ or } (y-b)\times\left(\frac{dy}{dx}\right)_{(a,b)} = -(x-a).$$

Remark

(i) If the tangent to a curve is horizontal at a point, then the derivative at that point is 0. Hence, at that point (x_1, y_1) the equation of the tangent is $y = y_1$ and equation of the normal is $x = x_1$.

(ii) If the tangent to a curve is vertical at a point, then the derivative exists and infinite (∞) at the point. Hence, at that point (x_1, y_1) the equation of the tangent is $x = x_1$ and the equation of the normal is $y = y_1$.

Example 7.11

Find the equations of tangent and normal to the curve $y = x^2 + 3x - 2$ at the point $(1,2)$.

Solution

We have, $\frac{dy}{dx} = 2x + 3$. Hence at $(1,2), \frac{dy}{dx} = 5$.

Therefore, the required equation of tangent is

$(y-2) = 5(x-1) \Rightarrow 5x - y - 3 = 0$.

The slope of the normal at the point $(1,2)$ is $-\frac{1}{5}$.

Therefore, the required equation of normal is

$(y-2) = -\frac{1}{5}(x-1) \Rightarrow x + 5y - 11 = 0$.

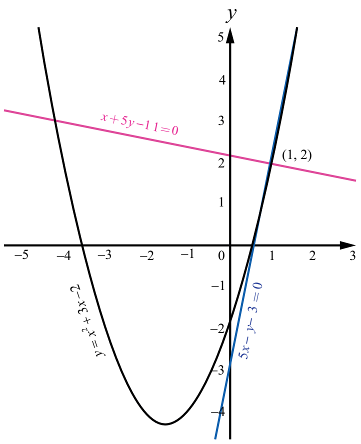


Fig.7.8

Example 7.12

Find the points on the curve $y = x^3 - 3x^2 + x - 2$ at which the tangent is parallel to the line $y = x$.

Solution

The slope of the line $y = x$ is 1. The tangent to the given curve will be parallel to the line, if the slope of the tangent to the curve at a point is also 1. Hence,

$$\frac{dy}{dx} = 3x^2 - 6x + 1 = 1$$

$$\text{which gives } 3x^2 - 6x = 0.$$

$$\text{Hence, } x = 0 \text{ and } x = 2.$$

Therefore, at (0, –2) and (2, –4) the tangent is parallel to the line $y = x$.

Example 7.13

Find the equation of the tangent and normal at any point to the Lissajous curve given by $x = 2\cos 3t$ and $y = 3\sin 2t, t \in \mathbb{R}$.

Solution

Observe that the given curve is neither a circle nor an ellipse. For your reference the curve is shown in Fig. 7.9.

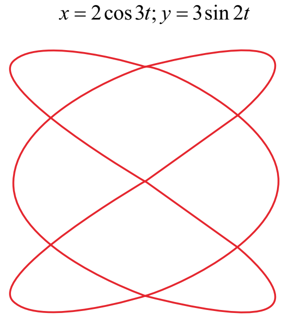
$$\text{Now, } \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= -\frac{6\cos 2t}{6\sin 3t} = -\frac{\cos 2t}{\sin 3t}.$$

Therefore, the tangent at any point is

$$y - 3\sin 2t = -\frac{\cos 2t}{\sin 3t}(x - 2\cos 3t)$$

That is, $x\cos 2t + y\sin 3t = 3\sin 2t\sin 3t + 2\cos 2t\cos 3t$.



Lissajous curve

Fig.7.9

The slope of the normal is the negative of the reciprocal of the tangent which in this case is $\dfrac{\sin 3t}{\cos 2t}$. Hence, the equation of the normal is

$$y-3\sin 2t=\frac{\sin 3t}{\cos 2t}(x-2\cos 3t).$$

That is, $x\sin 3t-y\cos 2t=2\sin 3t\cos 3t-3\sin 2t\cos 2t=\sin 6t-\dfrac{3}{2}\sin 4t$. ■

7.2.5 Angle between two curves

Definition 7.3

Angle between two curves, if they intersect, is defined as the acute angle between the tangent lines to those two curves at the point of intersection.

For the given curves, at the point of intersection using the slopes of the tangents, we can measure the acute angle between the two curves. Suppose $y=m_1x+c_1$ and $y=m_2x+c_2$ are two lines, then the acute angle θ between these lines is given by,

$$\tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right| \qquad \text{... (3)}$$

where m_1 and m_2 are finite.

Remark

(i) If the two curves are parallel at (x_1, y_1), then $m_1=m_2$.

(ii) If the two curves are perpendicular at (x_1, y_1) and if m_1 and m_2 exists and finite then $m_1m_2=-1$.

Example 7.14

Find the angle between $y=x^2$ and $y=(x-3)^2$.

Solution

Let us now find the point of intersection of the two given curves. Equating $x^2=(x-3)^2$ we get, $x=\dfrac{3}{2}$. Therefore, the point of intersection is $\left(\dfrac{3}{2},\dfrac{9}{4}\right)$. Let θ be the angle between the curves. The slopes of the curves are as follows :

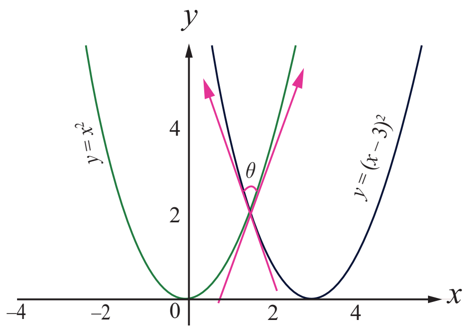


Fig.7.10

$$\text{For the curve } y = x^2,$$

$$\frac{dy}{dx} = 2x.$$

$$\text{Let } m_1=\frac{dy}{dx} \text{ at } \left(\frac{3}{2},\frac{9}{4}\right) = 3.$$

$$\text{For the curve } y = (x-3)^2,$$

$$\frac{dy}{dx} = 2(x-3).$$

$$\text{Let } m_2=\frac{dy}{dx} \text{ at } \left(\frac{3}{2},\frac{9}{4}\right) = -3.$$

Using (3), we get

$$\tan\theta = \left|\frac{3-(-3)}{1-9}\right| = \frac{3}{4}$$

$$\text{Hence, } \theta = \tan^{-1}\left(\frac{3}{4}\right).$$

Example 7.15

Find the angle between the curves $y = x^2$ and $x = y^2$ at their points of intersection (0,0) and (1,1).

Solution

Let us now find the slopes of the curves.

Let m_1 be the slope of the curve $y = x^2$,

$$\text{then } m_1 = \frac{dy}{dx} = 2x.$$

Let m_2 be the slope of the curve $x = y^2$,

$$\text{then } m_2 = \frac{dy}{dx} = \frac{1}{2y}.$$

Let θ_1 and θ_2 be the angles at (0,0) and (1,1) respectively.

At $(0,0)$, we come across the indeterminate form of $0\times\infty$ in the denominator of $\tan\theta_1 = \left|\dfrac{2x-\dfrac{1}{2y}}{1+(2x)\left(\dfrac{1}{2y}\right)}\right|$ and so we follow the limiting process.

$$\tan\theta_1 = \lim_{(x,y)\to(0,0)}\left|\frac{2x-\dfrac{1}{2y}}{1+(2x)\left(\dfrac{1}{2y}\right)}\right|$$

$$= \lim_{(x,y)\to(0,0)}\left|\frac{4xy-1}{2(y+x)}\right|$$

$$= \infty$$

$$\text{which gives } \theta_1 = \tan^{-1}(\infty) = \frac{\pi}{2}.$$

At $(1,1)$, $m_1 = 2$, $m_2 = \dfrac{1}{2}$

$$\tan\theta_2 = \left|\frac{2-\dfrac{1}{2}}{1+(2)\left(\dfrac{1}{2}\right)}\right|$$

$$= \frac{3}{4}$$

$$\text{which gives } \theta_2 = \tan^{-1}\left(\frac{3}{4}\right).$$

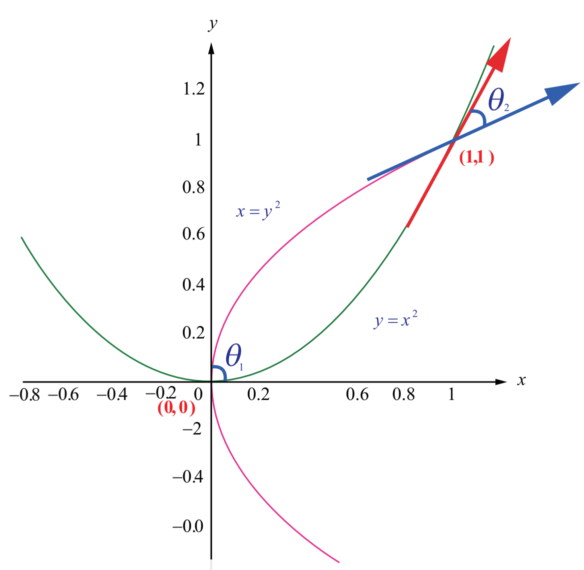


Fig.7.11

Example 7.16

Find the angle of intersection of the curve $y = \sin x$ with the positive x-axis.

Solution

When the curve $y = \sin x$ intersects the positive x-axis, $y = 0$ which gives, $x = n\pi$, $n = 1, 2, 3, \ldots$.

Now, $\frac{dy}{dx} = \cos x$. The slope at $x = n\pi$ are $\cos(n\pi) = (-1)^n$. Hence, the required angle of intersection is

$$\tan\theta = \left|\frac{(-1)^n - 0}{1 + (-1)^n(0)}\right| = 1 \qquad \forall\, n$$

Example 7.17

If the curves $ax^2 + by^2 = 1$ and $cx^2 + dy^2 = 1$ intersect each other orthogonally then,

$$\text{show that } \frac{1}{a} - \frac{1}{b} = \frac{1}{c} - \frac{1}{d}.$$

Solution

Let the two curves intersect at a point (x_0, y_0). This leads to $(a-c)x_0^2 + (b-d)y_0^2 = 0$.

Let us now find the slope of the curves at the point of intersection (x_0, y_0). The slopes of the curves are as follows :

$$\text{For the curve } ax^2 + by^2 = 1, \quad \frac{dy}{dx} = -\frac{ax}{by}.$$

$$\text{For the curve } cx^2 + dy^2 = 1, \quad \frac{dy}{dx} = -\frac{cx}{dy}.$$

Now, if two curves cut orthogonally, then the product of their slopes, at the point of intersection (x_0, y_0), is -1. Hence, if the above two curves cut orthogonally at (x_0, y_0) then

$$\left(-\frac{ax_0}{by_0}\right) \times \left(-\frac{cx_0}{dy_0}\right) = -1.$$

$$\text{That is, } acx_0^2 + bdy_0^2 = 0,$$

$$\text{together with } (a-c)x_0^2 + (b-d)y_0^2 = 0$$

$$\text{gives, } \frac{a-c}{ac} = \frac{b-d}{bd}.$$

$$\text{That is, } \frac{1}{c} - \frac{1}{a} = \frac{1}{d} - \frac{1}{b}.$$

$$\text{Hence, } \frac{1}{a} - \frac{1}{b} = \frac{1}{c} - \frac{1}{d}.$$

Remark

In the above example, the converse is also true. That is assuming the condition $\frac{1}{a} - \frac{1}{b} = \frac{1}{c} - \frac{1}{d}$ one can easily establish that the curves cut orthogonally.

Example 7.18

Prove that the ellipse $x^2+4y^2=8$ and the hyperbola $x^2-2y^2=4$ intersect orthogonally.

Solution

Let the point of intersection of the two curves be (a,b). Hence,

$$a^2+4b^2 = 8 \text{ and } a^2-2b^2=4 \quad \ldots (4)$$

It is enough to show that the product of the slopes of the two curves evaluated at (a,b) is -1.

Differentiation of $x^2+4y^2=8$ with respect x, gives

$$2x+8y\frac{dy}{dx} = 0$$

$$\text{Therefore } \frac{dy}{dx} = -\frac{x}{4y}.$$

$$\text{Then, } \frac{dy}{dx} \text{ at } (a,b) = m_1 = -\frac{a}{4b}.$$

Differentiation of $x^2-2y^2=4$ with respect to x, gives

$$2x-4y\frac{dy}{dx} = 0$$

$$\text{Therefore, } \frac{dy}{dx} = \frac{x}{2y}.$$

$$\text{Then } \frac{dy}{dx} \text{ at } (a,b) = m_2 = \frac{a}{2b}.$$

Therefore,

$$m_1 \times m_2 = \left(-\frac{a}{4b}\right)\times\left(\frac{a}{2b}\right) = -\frac{a^2}{8b^2} \quad \ldots (5)$$

Applying the ratio of proportions in (4), we get

$$\frac{a^2}{-16-16} = \frac{b^2}{-8+4} = \frac{1}{-2-4}.$$

Therefore $\frac{a^2}{b^2}=\frac{32}{4}=8$. Substituting in (5), we get $m_1\times m_2=-1$. Hence, the curves cut orthogonally. ■

EXERCISE 7.2

1. Find the slope of the tangent to the following curves at the respective given points.

   (i) $y=x^4+2x^2-x$ at $x=1$ (ii) $x=a\cos^3 t, y=b\sin^3 t$ at $t=\frac{\pi}{2}$.

2. Find the point on the curve $y=x^2-5x+4$ at which the tangent is parallel to the line $3x+y=7$.

3. Find the points on the curve $y=x^3-6x^2+x+3$ where the normal is parallel to the line $x+y=1729$.

4. Find the points on the curve $y^2-4xy=x^2+5$ for which the tangent is horizontal.

5. Find the tangent and normal to the following curves at the given points on the curve.

(i) $y = x^2 - x^4$ at $(1,0)$ (ii) $y = x^4 + 2e^x$ at $(0,2)$

(iii) $y = x\sin x$ at $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ (iv) $x = \cos t, y = 2\sin^2 t$ at $t = \frac{\pi}{3}$

6. Find the equations of the tangents to the curve $y = 1 + x^3$ for which the tangent is orthogonal with the line $x + 12y = 12$.

7. Find the equations of the tangents to the curve $y = \frac{x+1}{x-1}$ which are parallel to the line $x + 2y = 6$.

8. Find the equation of tangent and normal to the curve given by $x = 7\cos t$ and $y = 2\sin t, t \in \mathbb{R}$ at any point on the curve.

9. Find the angle between the rectangular hyperbola $xy = 2$ and the parabola $x^2 + 4y = 0$.

10. Show that the two curves $x^2 - y^2 = r^2$ and $xy = c^2$ where c, r are constants, cut orthogonally.

7.3 Mean Value Theorem

Mean value theorem establishes the existence of a point, in between two points, at which the tangent to the curve is parallel to the secant joining those two points of the curve. We start this section with the statement of the intermediate value theorem as follows :

Theorem 7.1 (Intermediate value theorem)

If f is continuous on a closed interval $[a,b]$, and c is any number between $f(a)$ and $f(b)$ inclusive, then there is at least one number x in the closed interval $[a,b]$, such that $f(x) = c$.

7.3.1 Rolle's Theorem

Theorem 7.2 (Rolle's Theorem)

Let $f(x)$ be continuous on a closed interval $[a,b]$ and differentiable on the open interval (a,b) If $f(a) = f(b)$, then there is at least one point $c \in (a,b)$ where $f'(c) = 0$.

Geometrically this means that if the tangent is moving along the curve starting at $x = a$ towards as in Fig 7.2 $x = b$ then there exists a $c \in (a,b)$ at which the tangent is parallel to the x-axis.

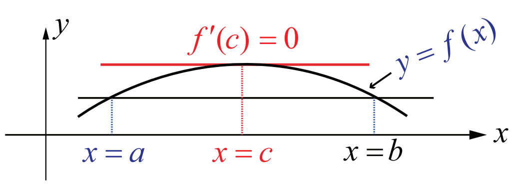


Fig.7.12

Example 7.19

Compute the value of 'c' satisfied by the Rolle's theorem for the function

$f(x) = x^2(1-x)^2, x \in [0,1]$.

Solution

Observe that, $f(0) = 0 = f(1), f(x)$ is continuous in the interval $[0,1]$ and is differentiable in $(0,1)$. Now,

$$f'(x) = 2x(1-x)(1-2x).$$

$$\text{Therefore, } f'(c) = 0 \text{ gives } c=0,1, \text{ and } \frac{1}{2}$$

$$\text{which} \Rightarrow c = \frac{1}{2} \in (0,1).$$

Example 7.20

Find the value in the interval $\left(\frac{1}{2},2\right)$ satisfied by the Rolle's theorem for the function $f(x)=x+\frac{1}{x}, x\in\left[\frac{1}{2},2\right]$.

Solution

We have, $f(x)$ is continuous in $\left[\frac{1}{2},2\right]$ and differentiable in $\left(\frac{1}{2},2\right)$ with $f\left(\frac{1}{2}\right)=\frac{5}{2}=f(2)$. By the Rolle's theorem there must exist a value $c\in\left(\frac{1}{2},2\right)$ such that,

$f'(c)=1-\frac{1}{c^2}=0 \Rightarrow c^2=1$ gives $c=\pm 1$. As $1\in\left(\frac{1}{2},2\right)$, we choose $c=1$.

Example 7.21

Compute the value of $'c'$ satisfied by Rolle's theorem for the function $f(x)=\log\left(\frac{x^2+6}{5x}\right)$ in the interval $[2,3]$.

Solution

Observe that, $f(2)=0=f(3)$ and $f(x)$ is continuous in the interval $[2,3]$ and differentiable in $(2,3)$. Now,

$$f'(x) = \frac{x^2-6}{x(x^2+6)}$$

$$\text{Therefore, } f'(c) = 0 \text{ gives}$$

$$\frac{c^2-6}{c(c^2+6)} = 0$$

$$\text{which implies } c = \pm\sqrt{6}$$

$$\text{Now } c = +\sqrt{6}\in(2,3).$$

Observe that $-\sqrt{6}\notin(2,3)$ and hence $c=+\sqrt{6}$ satisfies the Rolle's theorem.

Rolle's theorem can also be used to compute the number of roots of an algebraic equation in an interval without actually solving the equation.

Example 7.22

Without actually solving show that the equation $x^4+2x^3-2=0$ has only one real root in the interval $(0,1)$.

Solution

Let $f(x)=x^4+2x^3-2$. Then $f(x)$ is continuous in $[0,1]$ and differentiable in $(0,1)$. Now,

$$f'(x) = 4x^3+6x^2. \text{ If } f'(x)=0 \text{ then,}$$

$$2x^2(2x+3) = 0.$$

$$\text{Therefore, } x = 0, -\frac{3}{2} \text{ but } 0, -\frac{3}{2} \notin (0,1).$$

$$\text{Thus, } f'(x) > 0, \ \forall x \in (0,1).$$

Hence by the Rolle's theorem there do not exist $a, b \in (0,1)$ such that, $f(a) = 0 = f(b)$. Therefore the equation $f(x) = 0$ cannot have two roots in the interval $(0,1)$. But, $f(0) = -2 < 0$ and $f(1) = 1 > 0$ tells us the curve $y = f(x)$ crosses the x-axis between 0 and 1 only once by the Intermediate value theorem. Therefore the equation $x^4 + 2x^3 - 2 = 0$ has only one real root in the interval $(0,1)$. ■

As an application of the Rolle's theorem we have the following,

Example 7.23

Prove using the Rolle's theorem that between any two distinct real zeros of the polynomial

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

there is a zero of the polynomial

$$na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1.$$

Solution

Let $P(x) = a_n x^n + a_{n-1}x^{n-2} + \cdots + a_1 x + a_0$. Let $\alpha < \beta$ be two real zeros of $P(x)$. Therefore, $P(\alpha) = P(\beta) = 0$. Since $P(x)$ is continuous in $[\alpha, \beta]$ and differentiable in (α, β) by an application of Rolle's theorem there exists $\gamma \in (\alpha, \beta)$ such that $P'(\gamma) = 0$. Since,

$$P'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1$$

which completes the proof. ■

Example 7.24

Prove that there is a zero of the polynomial, $2x^3 - 9x^2 - 11x + 12$ in the interval $(2,7)$ given that 2 and 7 are the zeros of the polynomial $x^4 - 6x^3 - 11x^2 + 24x + 28$.

Solution

Applying the above example 7.23 with

$$P(x) = x^4 - 6x^3 - 11x^2 + 24x + 28, \ \alpha = 2, \ \beta = 7$$

and observing

$$\frac{P'(x)}{2} = 2x^3 - 9x^2 - 11x + 12 = Q(x), \text{ (say).}$$

This implies that there is a zero of the polynomial $Q(x)$ in the interval $(2,7)$. ■

For verification,

$$Q(2) = 16 - 36 - 22 + 12 = 28 - 58 = -30 < 0$$

$$Q(7) = 686 - 441 - 77 + 12 = 698 - 518 = 180 > 0$$

From this we may see that there is a zero of the polynomial $Q(x)$ in the interval $(2,7)$.

Remark

There are functions for which Rolle's theorem may not be applicable.

(1) For the function $f(x) = |x|, x \in [-1,1]$ Rolle's theorem is not applicable, even though $f(-1) = 1 = f(1)$ because $f(x)$ is not differentiable at $x = 0$.

(2) For the function,

$$f(x) = \begin{cases} 1 & \text{when } x = 0, \\ x & \text{when } 0 < x \le 1 \end{cases}.$$

even though $f(0) = f(1) = 1$, Rolle's theorem is not applicable because the function $f(x)$ is not continuous at $x = 0$.

(3) For the function $f(x) = \sin x,\ x \in \left[0, \frac{\pi}{2}\right]$ Rolle's theorem is not applicable, even though $f(x)$ is continuous in the closed interval $\left[0, \frac{\pi}{2}\right]$ and differentiable in the open interval $\left(0, \frac{\pi}{2}\right)$ because, $0 = f(0) \neq f\left(\frac{\pi}{2}\right) = 1$.

If $f(x)$ is continuous in the closed interval $[a,b]$ and differentiable in the open interval (a,b) and even if $f(a) \neq f(b)$ then the Rolle's theorem can be generalised as follows.

7.3.2 Lagrange's Mean Value Theorem

Theorem 7.3

Let $f(x)$ be continuous in a closed interval $[a,b]$ and differentiable in the open interval (a,b) (where $f(a), f(b)$ are not necessarily equal). Then there exist at least one point $c \in (a,b)$ such that,

$$f'(c) = \frac{f(b)-f(a)}{b-a} \qquad \dots (6)$$

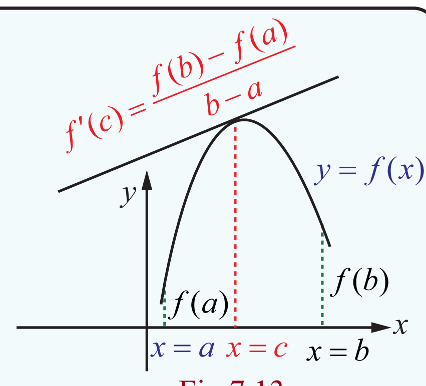


Fig.7.13

Remark

If $f(a) = f(b)$ then Lagrange's Mean Value Theorem gives the Rolle's theorem. It is also known as **rotated Rolle's Theorem**.

Remark

A physical meaning of the above theorem is the number $\frac{f(b)-f(a)}{b-a}$ can be thought of as the average rate of change in $f(x)$ over (a,b) and $f'(c)$ as an instantaneous change.

A geometrical meaning of the Lagrange's mean value theorem is that the instantaneous rate of change at some interior point is equal to the average rate of change over the entire interval. This is illustrated as follows :

If a car accelerating from zero takes just 8 seconds to travel 200 m, its average velocity for the 8 second interval is $\frac{200}{8} = 25$ m/s. The Mean Value Theorem says that at some point during the travel the speedometer must read exactly 90 km/h which is equal to 25 m/s.

Theorem 7.4

If $f(x)$ is continuous in closed interval $[a,b]$ and differentiable in open interval (a,b) and if $f'(x) > 0, \forall x \in (a,b)$, then for, $x_1, x_2 \in [a,b]$, such that $x_1 < x_2$ we have, $f(x_1) < f(x_2)$.

Proof

By the mean value theorem, there exists a $c \in (x_1, x_2) \subset (a,b)$ such that,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$$

Since $f'(c) > 0$, and $x_2 - x_1 > 0$ we have $f(x_2) - f(x_1) > 0$.

We conclude that, whenever $x_1 < x_2$, we have $f(x_1) < f(x_2)$.

Remark

If $f'(x) < 0, \forall x \in (a,b)$, then for, $x_1, x_2 \in [a,b]$, such that $x_1 < x_2$ we have, $f(x_1) > f(x_2)$. The proof is similar.

Example 7.25

Find the values in the interval $(1,2)$ of the mean value theorem satisfied by the function $f(x) = x - x^2$ for $1 \leq x \leq 2$.

Solution

$f(1) = 0$ and $f(2) = -2$. Clearly $f(x)$ is defined and differentiable in $1 < x < 2$. Therefore, by the Mean Value Theorem, there exists a $c \in (1,2)$ such that

$$f'(c) = \frac{f(2) - f(1)}{2 - 1} = 1 - 2c$$

That is, $1 - 2c = -2 \Rightarrow c = \frac{3}{2}$.

Geometrical meaning

Geometrically, the mean value theorem says the secant to the curve $y = f(x)$ between $x = a$ and $x = b$ is parallel to a tangent line of the curve, at some point $c \in (a,b)$.

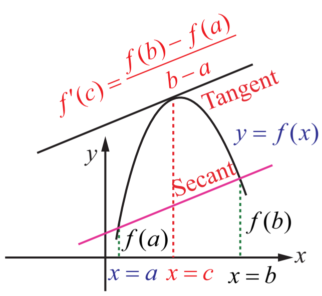


Fig.7.14

Consequences of Lagrange's Mean Value Theorem

There are three important consequences of MVT for derivatives.

(1) To determine the monotonicity of the given function (Theorem 7.4)

(2) If $f'(x) = 0$ for all x in (a,b), then f is constant on (a,b).

(3) If $f'(x) = g'(x)$ for all x, then $f(x) = g(x) + C$ for some constant C.

7.3.3 Applications

Example 7.26

A truck travels on a toll road with a speed limit of 80 km/hr. The truck completes a 164 km journey in 2 hours. At the end of the toll road the trucker is issued with a speed violation notice. Justify this using the Mean Value Theorem.

Solution

Let $f(t)$ be the distance travelled by the trucker in $'t'$ hours. This is a continuous function in $[0,2]$ and differentiable in $(0,2)$. Now, $f(0)=0$ and $f(2)=164$. By an application of the Mean Value Theorem, there exists a time c such that,

$$f'(c)=\frac{164-0}{2-0}=82>80.$$

Therefore at some point of time, during the travel in 2 hours the trucker must have travelled with a speed more than 80 km/hr which justifies the issuance of a speed violation notice. ■

Example 7.27

Suppose $f(x)$ is a differentiable function for all x with $f'(x)\leq 29$ and $f(2)=17$. What is the maximum value of $f(7)$?

Solution

By the mean value theorem we have, there exists $'c'\in(2,7)$ such that,

$$\frac{f(7)-f(2)}{7-2} = f'(c)\leq 29.$$

Hence, $f(7)\leq 5\times 29+17 = 162$

Therefore, the maximum value of $f(7)$ is 162. ■

Example 7.28

Prove, using mean value theorem, that

$$|\sin\alpha-\sin\beta|\leq|\alpha-\beta|,\ \alpha,\ \beta\in\mathbb{R}.$$

Solution

Let $f(x)=\sin x$ which is a differentiable function in any open interval. Consider an interval $[\alpha,\beta]$. Applying the mean value theorem there exists $c\in(\alpha,\beta)$ such that,

$$\frac{\sin\beta-\sin\alpha}{\beta-\alpha} = f'(c)=\cos(c)$$

Therefore, $$\left|\frac{\sin\alpha-\sin\beta}{\alpha-\beta}\right| = |\cos(c)|\leq 1$$

Hence, $|\sin\alpha-\sin\beta| \leq |\alpha-\beta|$. ■

Remark

If we take $\beta=0$ in the above problem, we get $|\sin\alpha|\leq|\alpha|$.

Example 7.29

A thermometer was taken from a freezer and placed in a boiling water. It took 22 seconds for the thermometer to raise from $-10°C$ to $100°C$. Show that the rate of change of temperature at some time t is $5°C$ per second.

Solution

Let $f(t)$ be the temperature at time t. By the mean value theorem, we have

$$\begin{aligned} f'(c) &= \frac{f(b)-f(a)}{b-a} \\ &= \frac{100-(-10)}{22} \\ &= \frac{110}{22} \\ &= 5°C \text{ per second.} \end{aligned}$$

Hence the instantaneous rate of change of temperature at some time t is $5°C$ per second. ■

EXERCISE 7.3

1. Explain why Rolle's theorem is not applicable to the following functions in the respective intervals.

   (i) $f(x)=\left|\frac{1}{x}\right|, x\in[-1,1]$ (ii) $f(x)=\tan x, x\in[0,\pi]$

   (iii) $f(x)=x-2\log x, x\in[2,7]$

2. Using the Rolle's theorem, determine the values of x at which the tangent is parallel to the x-axis for the following functions :

   (i) $f(x)=x^2-x,\ x\in[0,1]$ (ii) $f(x)=\frac{x^2-2x}{x+2},\ x\in[-1,6]$

   (iii) $f(x)=\sqrt{x}-\frac{x}{3},\ x\in[0,9]$

3. Explain why Lagrange's mean value theorem is not applicable to the following functions in the respective intervals :

   (i) $f(x)=\frac{x+1}{x}, x\in[-1,2]$ (ii) $f(x)=|3x+1|,\ x\in[-1,3]$

4. Using the Lagrange's mean value theorem determine the values of x at which the tangent is parallel to the secant line at the end points of the given interval:

   (i) $f(x)=x^3-3x+2,\ x\in[-2,2]$ (ii) $f(x)=(x-2)(x-7),\ x\in[3,11]$

5. Show that the value in the conclusion of the mean value theorem for

   (i) $f(x)=\frac{1}{x}$ on a closed interval of positive numbers $[a,b]$ is $\sqrt{ab}$

   (ii) $f(x)=Ax^2+Bx+C$ on any interval $[a,b]$ is $\frac{a+b}{2}$.

6. A race car driver is kilometer stone 20. If his speed never exceeds 150 km/hr, what is the maximum kilometer he can reach in the next two hours.

7. Suppose that for a function $f(x), f'(x)\le 1$ for all $1\le x\le 4$. Show that $f(4)-f(1)\le 3$.

8. Does there exist a differentiable function $f(x)$ such that $f(0)=-1, f(2)=4$ and $f'(x)\leq 2$ for all x. Justify your answer.

9. Show that there lies a point on the curve $f(x)=x(x+3)e^{-\frac{\pi}{2}}, -3\leq x\leq 0$ where tangent drawn is parallel to the x-axis.

10. Using mean value theorem prove that for, $a>0, b>0, |e^{-a}-e^{-b}|<|a-b|$.

7.4 Series Expansions

Taylor's series and Maclaurin's series expansion of a function which are infinitely differentiable.

Theorem 7.5

(a) Taylor's Series

Let $f(x)$ be a function infinitely differentiable at $x=a$. Then $f(x)$ can be expanded as a series, in an interval $(x-a, x+a)$, of the form

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n = f(a)+\frac{f'(a)}{1!}(x-a)+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\cdots.$$

(b) Maclaurin's series

If $a=0$, the expansion takes the form

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = f(0)+\frac{f'(0)}{1!}x+\cdots+\frac{f^{(n)}(0)}{n!}x^n+\cdots.$$

Proof

The series expansion of $f(x)$, in powers of $(x-a)$, be given by

$$f(x) = A_0+\sum_{n=1}^{\infty}A_n(x-a)^n \qquad \text{... (7)}$$

Substituting $x=a$ gives $A_0=f(a)$. Differentiation of (7) gives

$$f'(x) = 1!A_1+\sum_{n=2}^{\infty}nA_n(x-a)^{n-1} \qquad \text{... (8)}$$

Substituting $x=a$ gives $A_1=f'(a)$. Differentiation of (8) gives

$$f''(x) = 2!A_2+\sum_{n=3}^{\infty}n(n-1)A_n(x-a)^{n-2} \qquad \text{... (9)}$$

Substituting $x=a$ gives $A_2=\frac{f''(a)}{2!}$. Differentiation of (9) gives

$$f'''(x) = 3!A_3+\sum_{n=4}^{\infty}n(n-1)(n-2)A_n(x-a)^{n-3} \qquad \text{... (10)}$$

Differentiation of (10) $(k-3)$ times gives

$$f^{(k)}(x) = k!A_k+\sum_{n=k+1}^{\infty}n(n-1)...(n-k+1)A_n(x-a)^{n-k} \qquad \text{...(11)}$$

Substituting $x = a$ gives $A_k = \dfrac{f^{(k)}(a)}{k!}$ which completes the proof of the theorem.

In order to expand a function around a point say $x = a$, equivalently in powers of $(x - a)$ we need to differentiate the given function as many times as the required powers and evaluate at $x = a$. This will give the value for the coefficients of the required powers of $(x - a)$.

Example 7.30

Expand $\log(1+x)$ as a Maclaurin's series upto 4 non-zero terms for $-1 < x \le 1$.

Solution

Let $f(x) = \log(1+x)$, then the Maclaurin series of $f(x)$ is $f(x) = \sum_{n=0}^{n=\infty} a_n x^n$, where, $a_n = \dfrac{f^{(n)}(0)}{n!}$ various derivatives of the function $f(x)$ evaluated at $x = 0$ are given below:

Function and its derivatives	**$\log(1+x)$ and its derivatives**	**value at $x = 0$**
$f(x)$	$\log(1+x)$	0
$f'(x)$	$\dfrac{1}{1+x}$	1
$f''(x)$	$-\dfrac{1}{(1+x)^2}$	-1
$f'''(x)$	$\dfrac{2}{(1+x)^3}$	2
$f^{(iv)}(x)$	$-\dfrac{6}{(1+x)^4}$	-6

Table 7.2

Substituting the values and on simplification we get the required expansion of the function given by,

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad ; -1 < x \le 1.$$

Example 7.31

Expand $\tan x$ in ascending powers of x upto 5th power for $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$.

Solution

Let $f(x) = \tan x$. Then the Mclaurin series of $f(x)$ is

$$f(x) = \sum_{n=0}^{n=\infty} a_n x^n \text{, where, } a_n = \frac{f^{(n)}(0)}{n!}.$$

Various derivatives of the function $f(x)$ evaluated at $x = 0$ are given below:

Now,

$$f'(x) = \frac{d}{dx}(\tan x) = \sec^2(x)$$

$$f''(x) = \frac{d}{dx}(\sec^2(x)) = 2\sec x \cdot \sec x \cdot \tan x = 2\sec^2 x \cdot \tan x$$

$$f'''(x) = \frac{d}{dx}(2\sec^2(x) \cdot \tan x) = 2\sec^2(x) \cdot \sec^2 x + \tan x \cdot 4\sec x \cdot \sec x \cdot \tan x$$

$$= 2\sec^4 x + 4\sec^2 x \cdot \tan^2 x$$

$$f^{(iv)}(x) = 8\sec^3 x \cdot \sec x \cdot \tan x + 4\sec^2 x \cdot 2\tan x \cdot \sec^2 x + 8\sec x \cdot \sec x \cdot \tan x \cdot \tan^2 x$$

$$= 16\sec^4 x \tan x + 8\sec^2 x \cdot \tan^3 x$$

$$f^{(v)}(x) = 16\sec^4 x \cdot \sec^2 x + 64\sec^3 x \cdot \sec x \cdot \tan x \cdot \tan x + 8\sec^2 x \cdot 3\tan^2 x \cdot \sec^2 x$$
$$+16\sec x \cdot \sec x \cdot \tan x \cdot \tan^3 x$$
$$= 16\sec^6 x + 88\sec^4 x \cdot \tan^2 x + 16\sec^2 x \cdot \tan^4 x.$$

Function and its derivatives	**$\tan x$ and its derivatives**	**value at $x = 0$**
$f(x)$	$\tan x$	0
$f'(x)$	$\sec^2 x$	1
$f''(x)$	$2\sec^2 x \tan x$	0
$f'''(x)$	$2\sec^4 x + 4\sec^2 x \cdot \tan^2 x$	2
$f^{(iv)}(x)$	$16\sec^4 x \cdot \tan x + 8\sec^2 x \cdot \tan^3 x$	0
$f^{(v)}(x)$	$16\sec^6 x + 88\sec^4 x \cdot \tan^2 x + 16\sec^2 x \cdot \tan^4 x$	16

Table 7.3

Substituting the values and on simplification we get the required expansion of the function as

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots \quad ; -\frac{\pi}{2} < x < \frac{\pi}{2}.$$ ■

Example 7.32

Write the Taylor series expansion of $\frac{1}{x}$ about $x = 2$ by finding the first three non-zero terms.

Solution

Let $f(x) = \frac{1}{x}$, then the Taylor series of $f(x)$ is

$$f(x) = \sum_{n=0}^{n=\infty} a_n (x-2)^n, \text{ where } a_n = \frac{f^{(n)}(2)}{n!}.$$

Various derivatives of the function $f(x)$ evaluated at $x = 2$ are given below.

Functions and its derivatives	$\frac{1}{x}$ **and its derivatives**	**value at** $x = 2$
$f(x)$	$\frac{1}{x}$	$\frac{1}{2}$
$f'(x)$	$-\frac{1}{x^2}$	$-\frac{1}{4}$
$f''(x)$	$\frac{2}{x^3}$	$\frac{1}{4}$
$f'''(x)$	$-\frac{6}{x^4}$	$-\frac{3}{8}$

Table 7.4

Substituting these values, we get the required expansion of the function as

$$\frac{1}{x} = \frac{1}{2} - \frac{1}{4}\frac{(x-2)}{1!} + \frac{1}{4}\frac{(x-2)^2}{2!} - \frac{3}{8}\frac{(x-2)^3}{3!} + \cdots$$

which is, $$\frac{1}{x} = \frac{1}{2} - \frac{(x-2)}{4} + \frac{(x-2)^2}{8} - \frac{(x-2)^3}{16} + \cdots$$ ■

EXERCISE 7.4

1. Write the Maclaurin series expansion of the following functions:

   (i) e^x (ii) $\sin x$ (iii) $\cos x$

   (iv) $\log(1-x)$; $-1 \le x < 1$ (v) $\tan^{-1}(x)$; $-1 \le x \le 1$ (vi) $\cos^2 x$

2. Write down the Taylor series expansion, of the function $\log x$ about $x = 1$ upto three non-zero terms for $x > 0$.
3. Expand $\sin x$ in ascending powers $x - \frac{\pi}{4}$ upto three non-zero terms.
4. Expand the polynomial $f(x) = x^2 - 3x + 2$ in powers of $x - 1$.

7.5 Indeterminate Forms

In this section, we shall discuss various "indeterminate forms" and methods of evaluating the limits when we come across them.

7.5.1 A Limit Process

While computing the limits

$$\lim_{x\to\alpha} R(x)$$

of certain functions $R(x)$, we may come across the following situations like,

$$\frac{0}{0},\ \frac{\infty}{\infty},\ 0\times\infty,\ \infty-\infty,\ 1^{\infty},\ 0^{0},\ \infty^{0}.$$

We say that they have the form of a number. But values cannot be assigned to them in a way that is consistent with the usual rules of addition and mutiplication of numbers. We call these expressions indeterminate forms. Although they are not numbers, these indeterminate forms play a useful role in the limiting behaviour of a function.

John (Johann) Bernoulli discovered a rule using derivatives to compute the limits of fractions whose numerators and denominators both approach zero or ∞. The rule is known today as l'Hôpital's Rule (pronounced as Lho pi tal Rule), named after Guillaume de l'Hospital's, a French nobleman who wrote the earliest introductory differential calculus text, where the rule first appeared in print.

7.5.2 The l'Hôpital's Rule

Suppose $f(x)$ and $g(x)$ are differentiable functions and $g'(x) \neq 0$ with

$$\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)\text{. Then } \lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

$$\lim_{x\to a} f(x) = \pm\infty = \lim_{x\to a} g(x)\text{. Then } \lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

7.5.3 Indeterminate forms $\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty, \infty-\infty$

Example 7.33

Evaluate : $\lim_{x\to 1}\left(\frac{x^2-3x+2}{x^2-4x+3}\right)$.

Solution

If we put directly $x=1$ we observe that the given function is in an indeterminate form $\frac{0}{0}$. As the numerator and the denominator functions are polynomials of degree 2 they both are differentiable. Hence, by an application of the l'Hôpital Rule, we get

$$\lim_{x\to 1}\left(\frac{x^2-3x+2}{x^2-4x+3}\right) = \lim_{x\to 1}\left(\frac{2x-3}{2x-4}\right)$$
$$= \frac{1}{2}.$$

■

Note that this limit may also be evaluated through the factorization of the numerator and denominator as $\frac{x^2-3x+2}{x^2-4x+3} = \frac{(x-1)(x-2)}{(x-1)(x-3)}$.

Example 7.34

Compute the limit $\lim_{x\to a}\left(\frac{x^n-a^n}{x-a}\right)$.

Solution

If we put directly $x=a$ we observe that the given function is in an indeterminate form $\frac{0}{0}$. As the numerator and the denominator functions are polynomials they both are differentiable.

Hence by an application of the l'Hôpital Rule we get,

$$\lim_{x\to a}\left(\frac{x^n-a^n}{x-a}\right) = \lim_{x\to a}\left(\frac{n\times x^{n-1}}{1}\right)$$

$$= n\times a^{n-1}.$$

Example 7.35

Evaluate the limit $\lim_{x\to 0}\left(\frac{\sin mx}{x}\right)$.

Solution

If we directly substitute $x=0$ we get an indeterminate form $\frac{0}{0}$ and hence we apply the l'Hôpital's rule to evaluate the limit as,

$$\lim_{x\to 0}\left(\frac{\sin mx}{x}\right) = \lim_{x\to 0}\left(\frac{m\times\cos mx}{1}\right)$$

$$= m$$

The next example tells that the limit does not exist.

Example 7.36

Evaluate the limit $\lim_{x\to 0}\left(\frac{\sin x}{x^2}\right)$.

Solution

If we directly substitute $x=0$ we get an indeterminate form $\frac{0}{0}$ and hence we apply the l'Hôpital's rule to evaluate the limit as,

$$\lim_{x\to 0^+}\left(\frac{\sin x}{x^2}\right) = \lim_{x\to 0^+}\left(\frac{\cos x}{2x}\right) = \infty$$

$$\lim_{x\to 0^-}\left(\frac{\sin x}{x^2}\right) = \lim_{x\to 0^-}\left(\frac{\cos x}{2x}\right) = -\infty$$

As the left limit and the right limit are not the same we conclude that the limit does not exist.

Remark

One may be tempted to use the l'Hôpital's rule once again in $\lim_{x\to 0^+}\left(\frac{\cos x}{2x}\right)$ to conclude

$$\lim_{x\to 0^+}\left(\frac{\cos x}{2x}\right) = \lim_{x\to 0^+}\left(\frac{-\sin x}{2}\right) = 0 .$$

which is not true because it was not an indeterminate form.

Example 7.37

If $\lim_{\theta\to 0}\left(\frac{1-\cos m\theta}{1-\cos n\theta}\right) = 1$, then prove that $m=\pm n$.

Solution

As this is an indeterminate form $\left(\frac{0}{0}\right)$, using the l'Hôpital's Rule

$$\lim_{\theta\to 0}\left(\frac{1-\cos m\theta}{1-\cos n\theta}\right) = \lim_{\theta\to 0}\left(\frac{m\sin m\theta}{n\sin n\theta}\right)$$

Now using the example 7.35, we have

$$\lim_{\theta\to 0}\frac{m}{n}\times\left(\frac{\frac{\sin m\theta}{\theta}}{\frac{\sin n\theta}{\theta}}\right) = \frac{m^2}{n^2}$$

$$\text{Therefore } m^2 = n^2$$

$$\text{That is } m = \pm n.$$

Example 7.38

Evaluate : $\lim\limits_{x\to 1^-}\left(\dfrac{\log(1-x)}{\cot(\pi x)}\right)$.

Solution

This is an indeterminate form $\dfrac{\infty}{\infty}$ and hence we use the l'Hôpital's Rule to evaluate the limit.

$$\text{Thus, } \lim_{x\to 1^-}\frac{\log(1-x)}{\cot(\pi x)} = \lim_{x\to 1^-}\left(\frac{-\frac{1}{1-x}}{-\pi\operatorname{cosec}^2(\pi x)}\right)\left(\frac{\infty}{\infty}\text{ form}\right)$$

On simplication, we get

$$= \lim_{x\to 1^-}\left(\frac{\sin^2(\pi x)}{\pi(1-x)}\right). \quad \left(\frac{0}{0}\text{ form}\right)$$

again applying the l'Hôpital Rule, we get

$$= \lim_{x\to 1^-}\left(\frac{2\pi\sin(\pi x)\cdot\cos(\pi x)}{-\pi}\right)$$

$$= \lim_{x\to 1^-}\left(-2\sin(\pi x)\cdot\cos(\pi x)\right)$$

$$= 0.$$

Example 7.39

Evaluate : $\lim\limits_{x\to 0^+}\left(\dfrac{1}{x}-\dfrac{1}{e^x-1}\right)$.

Solution

This is an indeterminate of the form $\infty-\infty$. To evaluate this limit we first simplify and bring it in the form $\left(\dfrac{0}{0}\right)$ and applying the l'Hôpital Rule, we get

$$\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{e^x-1}\right) = \lim_{x\to 0^+}\left(\frac{e^x-x-1}{x(e^x-1)}\right). \quad \left(\frac{0}{0}\text{ form}\right)$$

Now,

$$\lim_{x\to 0^+}\left(\frac{e^x-x-1}{x(e^x-1)}\right) = \lim_{x\to 0^+}\left(\frac{e^x-1}{xe^x+e^x-1}\right) \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to 0^+}\left(\frac{e^x}{xe^x+2e^x}\right) = \frac{1}{2}.$$

Example 7.40

Evaluate : $\lim\limits_{x \to 0^+} x \log x$.

Solution

This is an indeterminate of the form $(0 \times \infty)$. To evaluate this limit, we first simplify and bring it to the form $\left(\frac{\infty}{\infty}\right)$ and apply l'Hôpital Rule. Thus, we get

$$\lim_{x \to 0^+} x \log x = \lim_{x \to 0^+} \left(\frac{\log x}{\frac{1}{x}} \right) \quad \left(\frac{\infty}{\infty} \text{ form} \right)$$

$$= \lim_{x \to 0^+} \left(\frac{\frac{1}{x}}{-\frac{1}{x^2}} \right) = \lim_{x \to 0^+} (-x) = 0.$$

Example 7.41

Evaluate : $\lim\limits_{x \to \infty} \left(\dfrac{x^2 + 17x + 29}{x^4} \right)$.

Solution

This is an indeterminate of the form $\left(\frac{\infty}{\infty}\right)$. To evaluate this limit, we apply l'Hôpital Rule.

Then, we have $$\lim_{x \to \infty} \left(\frac{x^2 + 17x + 29}{x^4} \right) = \lim_{x \to \infty} \left(\frac{2x + 17}{4x^3} \right)$$

$$= \lim_{x \to \infty} \left(\frac{2}{12x^2} \right) = 0.$$

Example 7.42

Evaluate : $\lim\limits_{x \to \infty} \left(\dfrac{e^x}{x^m} \right), m \in N$.

Solution

This is an indeterminate of the form $\left(\frac{\infty}{\infty}\right)$.

To evaluate this limit, we apply l'Hôpital Rule m times.

Thus, we have $$\lim_{x \to \infty} \frac{e^x}{x^m} = \lim_{x \to \infty} \frac{e^x}{m!}$$

$$= \infty.$$

7.5.4 Indeterminate forms $0^0, 1^\infty$ and ∞^0

In order to evaluate the indeterminate forms like this, we shall first state the theorem on the limit of a composite function.

Theorem 7.6

Let $\lim\limits_{x \to \alpha} g(x)$ exist and let it be L and let $f(x)$ be a continuous function at $x = L$. Then,

$$\lim_{x \to \alpha} f(g(x)) = f\left(\lim_{x \to \alpha} g(x) \right).$$

The evaluation procedure for evaluating the limits

(1) Let $A = \lim_{x \to a} g(x)$. Then taking logarithm, with the assumption that $A > 0$ to ensure the continuity of the logarithm function, we get $\log A = \lim_{x \to a} \log(g(x))$. Therefore using the above theorem with $f(x) = \log x$ we have the limit

$$\lim_{x \to a} \log(g(x)) = \log\left(\lim_{x \to a} g(x)\right).$$

(2) We have the limit $\lim_{x \to a} \log(g(x))$ into either $\left(\frac{0}{0}\right)$ or $\left(\frac{\infty}{\infty}\right)$ form evaluate it using l'Hôpital Rule.

(3) Let that evaluated limit be say α. Then the required limit is e^{α}.

Example 7.43

Using the l'Hôpital Rule, prove that $\lim_{x \to 0^+} (1+x)^{\frac{1}{x}} = e$.

Solution

This is an indeterminate of the form 1^{∞}. Let $g(x) = (1+x)^{\frac{1}{x}}$. Taking the logarithm, we get

$$\log g(x) = \frac{\log(1+x)}{x}$$

$$\lim_{x \to 0^+} \log(g(x)) = \lim_{x \to 0^+} \left(\frac{\log(1+x)}{x}\right) \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0^+} \left(\frac{\frac{1}{1+x}}{1}\right) \qquad \text{(by l'Hôpital Rule)}$$

$$= 1.$$

$$\text{But, } \lim_{x \to 0^+} \log g(x) = \log\left(\lim_{x \to 0^+} g(x)\right)$$

$$\text{Therefore, } \log\left(\lim_{x \to 0^+} g(x)\right) = 1.$$

Hence by exponentiating, we get $\lim_{x \to 0^+} g(x) = e$. ■

Example 7.44

Evaluate : $\lim_{x \to \infty} (1+2x)^{\frac{1}{2\log x}}$.

Solution

This is an indeterminate of the form ∞^{0}.

$$\text{Let } g(x) = (1+2x)^{\frac{1}{2\log x}}.$$

Taking the logarithm, we get

$$\log g(x) = \frac{\log(1+2x)}{2\log x}$$

$$\lim_{x\to\infty}\log g(x) = \lim_{x\to\infty}\left(\frac{\log(1+2x)}{2\log x}\right)\quad\left(\frac{\infty}{\infty}\text{ form}\right)$$

$$= \lim_{x\to\infty}\left(\frac{\frac{2}{1+2x}}{\frac{2}{x}}\right)\quad\text{(by l'Hôpital Rule)}$$

$$= \lim_{x\to\infty}\left(\frac{x}{1+2x}\right)\quad\left(\frac{\infty}{\infty}\text{ form}\right)$$

$$= \lim_{x\to\infty}\left(\frac{1}{2}\right) = \frac{1}{2}\quad\text{but,}$$

$$\lim_{x\to\infty}\log g(x) = \log\left(\lim_{x\to\infty} g(x)\right).$$

Hence by exponentiating, we get the required limit as $\sqrt{e}$.

Example 7.45

Evaluate : $\lim_{x\to 1} x^{\frac{1}{1-x}}$.

Solution

Let $g(x) = x^{\frac{1}{1-x}}$. This is an indeterminate of the form 1^{∞} when $x\to 1$. Taking the logarithm,

$$\log g(x) = \frac{\log x}{1-x}.$$

$$\text{Therefore, } \lim_{x\to 1}\log g(x) = \lim_{x\to 1}\left(\frac{\log x}{1-x}\right)\quad\left(\frac{0}{0}\text{ form}\right).$$

An application of l'Hôpital rule, gives

$$\lim_{x\to 1}\left(\frac{\frac{1}{x}}{-1}\right) = -1.$$

$$\text{But, } \lim_{x\to 1}\log g(x) = \log\left(\lim_{x\to 1} g(x)\right).$$

Hence on exponentiating, we get

$$\lim_{x\to 1} x^{\frac{1}{1-x}} = e^{-1} = \frac{1}{e}.$$

EXERCISE 7.5

Evaluate the following limits, if necessary use l'Hôpital Rule :

1. $\lim_{x\to 0}\frac{1-\cos x}{x^2}$
2. $\lim_{x\to\infty}\frac{2x^2-3}{x^2-5x+3}$
3. $\lim_{x\to\infty}\frac{x}{\log x}$
4. $\lim_{x\to\frac{\pi}{2}^-}\frac{\sec x}{\tan x}$
5. $\lim_{x\to\infty} e^{-x}\sqrt{x}$
6. $\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)$
7. $\lim_{x\to 1^+}\left(\frac{2}{x^2-1}-\frac{x}{x-1}\right)$
8. $\lim_{x\to 0^+} x^x$
9. $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$

10. $\lim\limits_{x \to \frac{\pi}{2}} (\sin x)^{\tan x}$ 11. $\lim\limits_{x \to 0^+} (\cos x)^{\frac{1}{x^2}}$

12. If an initial amount A_0 of money is invested at an interest rate r compounded n times a year, the value of the investment after t years is $A = A_0\left(1+\frac{r}{n}\right)^{nt}$. If the interest is compounded continuously, (that is as $n \to \infty$), show that the amount after t years is $A = A_0 e^{rt}$

7.6 Applications of First Derivative

Using the first derivative we can test a function $f(x)$ for its monotonicity (increasing or decreasing), focusing on a particular point in its domain and the local extrema (maxima or minima) on a domain.

7.6.1 Monotonicity of functions

Monotonicity of functions are its behaviour of increasing or decreasing.

Definition 7.4

A function $f(x)$ is said to be an increasing function in an interval I if $a < b \Rightarrow f(a) \le f(b), \forall a, b \in I$.

Definition 7.5

A function $f(x)$ is said to be a decreasing function in an interval I if $a < b \Rightarrow f(a) \ge f(b), \forall a, b \in I$.

The function $f(x) = x$ is an increasing function in the entire real line, whereas the function $f(x) = -x$ is a decreasing function in the entire real line. In general, a given function may be increasing in some interval and decreasing in some other interval, say for instance, the function $f(x) = |x|$ is decreasing in $(-\infty, 0]$ and is increasing in $[0, \infty)$. These functions are simple to observe for their monotonicity. But given an arbitrary function how we determine its monotonicity in an interval of a real line? That is where following theorem (stated without proof) will be useful.

Theorem 7.7

If the function $f(x)$ is differentiable in an open interval (a, b) then we say,

(1) if
$$\frac{d}{dx}(f(x)) \ge 0, \forall x \in (a, b), \quad \text{... (1)}$$
then $f(x)$ is increasing in the interval (a, b),

(2) if
$$\frac{d}{dx}(f(x)) > 0, \forall x \in (a, b), \quad \text{... (2)}$$
then $f(x)$ is strictly increasing in the interval (a, b).

The proof of the above can be observed from Theorem 7.3.

(3) $f(x)$ is decreasing in the interval (a, b) if

$$\frac{d}{dx}(f(x)) \le 0, \forall x \in (a,b). \qquad \text{...(3)}$$

(4) $f(x)$ is strictly decreasing in the interval (a,b) if

$$\frac{d}{dx}(f(x)) < 0, \forall x \in (a,b). \qquad \text{... (4)}$$

Remark

It is very important to note the following fact. *It is* false *to say that if a differentiable function* $f(x)$ *on* I *is strictly increasing on* I*, then* $f'(x) > 0$ *for all* $x \in I$. For instance, consider $y = x^3$, $x \in (-\infty, \infty)$. It is strictly increasing on $(-\infty, \infty)$. To prove this, let $a > b$. Then we have to prove that $f(a) > f(b)$. For this purpose, we have to prove $a^3 - b^3 > 0$.

Now,

$$\begin{aligned} a^3 - b^3 &= (a-b)(a^2 + ab + b^2) \\ &= (a-b)\frac{1}{2}(2a^2 + 2ab + 2b^2) \\ &= (a-b)\frac{1}{2}\left((a+b)^2 + a^2 + b^2\right) \\ &> 0 \text{ since } a - b > 0 \text{ and other terms inside the bracket are } > 0. \end{aligned}$$

Hence it is clear that the quadratic expression is always positive (it is equal to zero only if $a = b = 0$, which contradicts the condition $a < b$). Therefore the function is $y = x^3$ is strictly increasing in $(-\infty, \infty)$. But $f'(x) = 3x^2$ which is equal to zero at $x = 0$.

Definition 7.6

A stationary point $(x_0, f(x_0))$ of a differentiable function $f(x)$ is where $f'(x_0) = 0$.

Definition 7.7

A critical point $(x_0, f(x_0))$ of a function $f(x)$ is where $f'(x_0) = 0$ or does not exist.

Note

We State that if (x,y) is a Stationary point or Critical Point of f where x from the domain of f is called Stationary number or Critical number

Every stationary point is a critical point however all critical points need not be stationary points. As an example, the function $f(x) = |x - 17|$ has a critical point at $(17, 0)$ but $(17, 0)$ is not a stationary point as the function has no derivative at $x = 17$.

Example 7.46

Prove that the function $f(x) = x^2 + 2$ is strictly increasing in the interval $(2, 7)$ and strictly decreasing in the interval $(-2, 0)$.

Solution

We have,

$$\begin{aligned} f'(x) &= 2x > 0, \forall x \in (2,7) \text{ and} \\ f'(x) &= 2x < 0, \forall x \in (-2,0) \end{aligned}$$

and hence the proof is completed. ■

Example 7.47

Prove that the function $f(x) = x^2 - 2x - 3$ is strictly increasing in $(2, \infty)$.

Solution

Since $f(x) = x^2 - 2x - 3$, $f'(x) = 2x - 2 > 0 \; \forall \, x \in (2, \infty)$. Hence $f(x)$ is strictly increasing in $(2, \infty)$. ■

7.6.2 Absolute maxima and minima

The absolute maxima and absolute minima are referred to describing the largest and smallest values of a function on an interval.

Definition 7.8

Let x_0 be a number in the domain D of a function $f(x)$. Then $f(x_0)$ is the absolute maximum value of $f(x)$ on D, if $f(x_0) \geq f(x) \forall x \in D$ and $f(x_0)$ is the absolute minimum value of $f(x)$ on D if $f(x_0) \leq f(x) \forall x \in D$.

In general, there is no guarantee that a function will actually have an absolute maximum or absolute minimum on a given interval. The following figures show that a continuous function may or may not have absolute maxima or minima on an infinite interval or on a finite open interval.

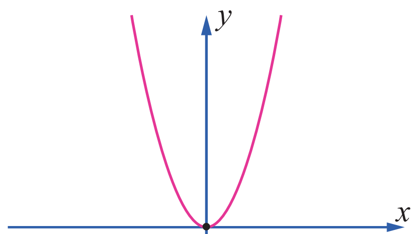


Fig. 7.15

$f(x)$ has an absolute minimum but no absolute maximum on $(-\infty, \infty)$

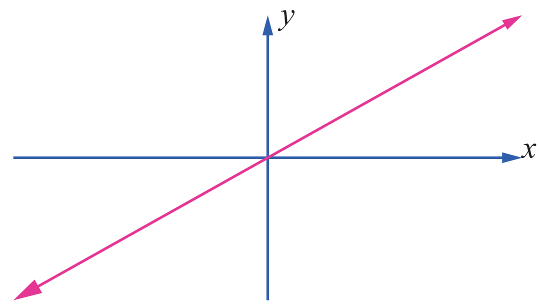


Fig. 7.16

$f(x)$ has no absolute extrema on $(-\infty, \infty)$.

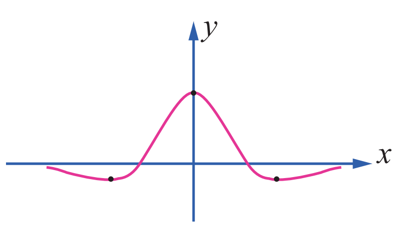


Fig. 7.17

$f(x)$ has an absolute maximum and an absolute minimum on $(-\infty, \infty)$

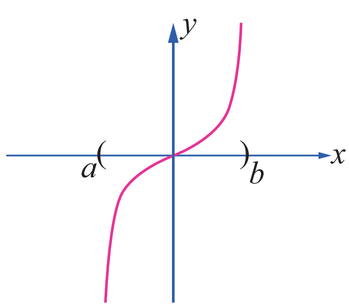


Fig. 7.18

$f(x)$ has no absolute extrema on (a, b).

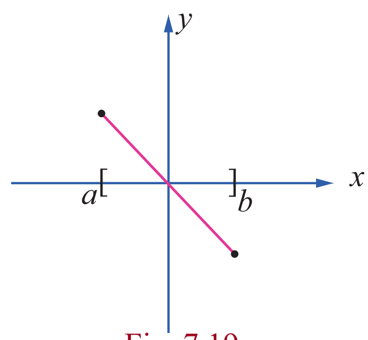


Fig. 7.19

$f(x)$ has an absolute maximum and an absolute minimum on $[a, b]$

However, the following theorem shows that a continuous function must have both an absolute maximum and an absolute minimum on every closed interval.

Theorem 7.8 (Extreme Value Theorem)

If $f(x)$ is continuous on a closed interval $[a,b]$, then f has both an absolute maximum and an absolute minimum on $[a,b]$.

The absolute extrema of $f(x)$ occur either at the endpoints of closed interval $[a,b]$ or inside the open interval (a,b).If the absolute extrema occurs inside, then it must occur at critical numbers of $f(x)$. Thus, we can use the following procedure to find the absolute extrema of a continuous function on closed interval $[a,b]$.

A procedure for finding the absolute extrema of a continuous function $f(x)$ on closed interval $[a,b]$.

Step 1 : Find the critical numbers of $f(x)$ in (a,b)

Step 2 : Evaluate $f(x)$ at all the critical numbers and at the endpoints a and b

Step 3 : The largest and the smallest of the values in step 2 is the absolute maximum and absolute minimum of $f(x)$ respectively on the closed interval $[a,b]$.

Example 7.48

Find the absolute maximum and absolute minimum values of the function $f(x)=2x^3+3x^2-12x$ on $[-3,2]$

Solution

Differentiating the given function, we get

$$\begin{aligned} f'(x) &= 6x^2+6x-12 \\ &= 6(x^2+x-2) \\ f'(x) &= 6(x+2)(x-1) \end{aligned}$$

Thus, $f'(x)=0 \Rightarrow x=-2,1 \in (-3,2)$.

Therefore, the critical numbers are $x=-2,1$. Evaluating $f(x)$ at the endpoints $x=-3,2$ and at critical numbers $x=-2,1$, we get $f(-3)=9$, $f(2)=4$, $f(-2)=20$ and $f(1)=-7$.

From these values, the absolute maximum is 20 which occurs at $x=-2$, and the absolute minimum is -7 which occurs at $x=1$. ■

Example 7.49

Find the absolute extrema of the function $f(x)=3\cos x$ on the closed interval $[0,2\pi]$.

Solution

Differentiating the given function, we get $f'(x)=-3\sin x$.

Thus, $f'(x)=0 \Rightarrow \sin x=0 \Rightarrow x=\pi \in (0,2\pi)$. Evaluating $f(x)$ at the endpoints $x=0, 2\pi$ and at critical number $x=\pi$, we get $f(0)=3$, $f(2\pi)=3$, and $f(\pi)=-3$.

From these values, the absolute maximum is 3 which occurs at $x=0,\ 2\pi$, and the absolute minimum is -3 which occurs at $x=\pi$. ■

7.6.3 Relative Extrema on an Interval

A function $f(x)$ is said to have a relative maximum at x_0, if there is an open interval containing x_0 on which $f(x_0)$ is the largest value. Similarly, $f(x)$ is said to have a relative minimum at x_0, if there is an open interval containing x_0 on which $f(x_0)$ is the smallest value.

A relative maximum need not be the largest value on the entire domain, while a relative minimum need not be the smallest value on the entire domain. Therefore, there may be more than one relative maximum or relative minimum on the entire domain.

A relative extrema of a function is the extreme values (maximum or minimum) of the functions among all the evaluated values of $f(x), \forall x \in I \subset D$ where I may be open or closed. Usually the local extreme value of a function is attained at a critical point. Note that, a function may have a critical point at $x = c$ without having a local extreme value there. For instance, both of the functions $y = x^3$ and $y = x^{\frac{1}{3}}$ have critical points at the origin, but neither function has a local extreme value at the origin.

Theorem 7.9 (Fermat)

If $f(x)$ has a relative extrema at $x = c$ then c is a critical number. Invariably there will be critical numbers of the function obtained as solutions of the equation $f'(x) = 0$ or as values of x at which $f'(x)$ does not exist.

7.6.4 Extrema using First Derivative Test

After we have determined the intervals on which a function is increasing or decreasing, it is not difficult to locate the relative extrema of the function. The location or points at which the relative extrema occurs for a given function $f(x)$ can be observed through the graph $y = f(x)$. However to find the exact point and the value of the extrema of functions we need to use certain test on functions. One such test is the first derivative test, which is stated in the following theorem.

Theorem 7.10 (First Derivative Test)

Let $(c, f(c))$ be a critical point of function $f(x)$ that is continuous on an open interval I containing c. If $f(x)$ is differentiable on the interval, except possibly at c, then $f(c)$ can be classified as follows: (when moving across the interval I from left to right)

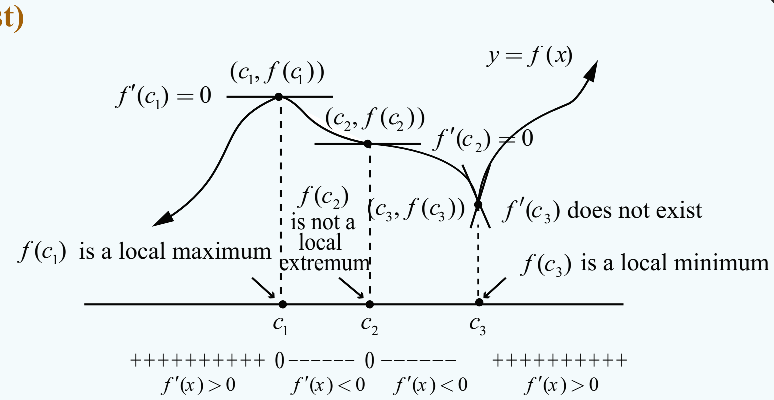


Fig. 7.20

(i) If $f'(x)$ changes from negative to positive at c, then $f(x)$ has a local minimum $f(c)$.

(ii) If $f'(x)$ changes from positive to negative at c, then $f(x)$ has a local maximum $f(c)$.

(iii) If $f'(x)$ is positive on both sides of c or negative on both sides of c, then $f(c)$ is neither a local minimum nor a local maximum.

Example 7.50

Find the intervals of monotonicity and hence find the local extrema for the function $f(x) = x^2 - 4x + 4$.

E3I2Y4

Solution

We have,

$$f(x) = (x-2)^2 \text{, then}$$
$$f'(x) = 2(x-2) = 0 \text{ gives } x = 2 .$$

The intervals of monotonicity are $(-\infty, 2)$ and $(2, \infty)$. Since $f'(x) < 0$, for $x \in (-\infty, 2)$ the function $f(x)$ is strictly decreasing on $(-\infty, 2)$. As $f'(x) > 0$, for $x \in (2, \infty)$ the function $f(x)$ is strictly increasing on $(2, \infty)$. Because $f'(x)$ changes its sign from negative to positive when passing through $x = 2$ for the function $f(x)$, it has a local minimum at $x = 2$. The local minimum value is $f(2) = 0$. ■

Example 7.51

Find the intervals of monotonicity and hence find the local extrema for the function $f(x) = x^{\frac{2}{3}}$.

Solution

We have, $f(x) = x^{\frac{2}{3}}$, then $f'(x) = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3x^{\frac{1}{3}}}$. $f'(x) \neq 0 \;\; \forall x \in \mathbb{R}$ and $f'(x)$ does not exist at $x = 0$. Therefore, there are no stationary points but there is a critical point at $x = 0$.

Interval	(-∞, 0)	(0, ∞)
Sign of $f'(x)$	$-$	$+$
Monotonicity	strictly decreasing	strictly increasing

Table 7.5

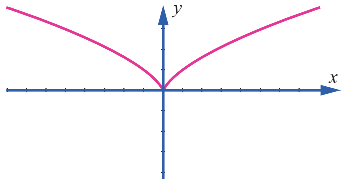


Fig.7.21

Because $f'(x)$ changes its sign from negative to positive when passing through $x = 0$ for the function $f(x)$, it has a local minimum at $x = 0$. The local minimum value is $f(0) = 0$. Note that here the local minimum occurs at a critical point which is not a stationary point. ■

Example 7.52

Prove that the function $f(x) = x - \sin x$ is increasing on the real line. Also discuss for the existence of local extrema.

Solution

Since $f'(x) = 1 - \cos x \geq 0$ and zero at the points $x = 2n\pi,\ n \in \mathbb{Z}$ and hence the function is increasing on the real line.

Since there is no sign change in $f'(x)$ when passing through $x = 2n\pi,\ n \in \mathbb{Z}$ by the first derivative test there is no local extrema. ■

Example 7.53

Discuss the monotonicity and local extrema of the function

$$f(x) = \log(1+x) - \frac{x}{1+x}, x > -1 \text{ and hence find the domain where, } \log(1+x) > \frac{x}{1+x}.$$

Solution

We have,

$$f(x) = \log(1+x) - \frac{x}{1+x}$$

$$\text{Therefore,} \quad f'(x) = \frac{1}{1+x} - \frac{1}{(1+x)^2} = \frac{x}{(1+x)^2}.$$

Hence,

$$f'(x) \text{ is } \begin{cases} <0 & \text{when } -1<x<0 \\ =0 & \text{when } x=0 \\ >0 & \text{when } x>0 \end{cases}$$

Therefore $f(x)$ is strictly increasing for $x>0$ and strictly decreasing for $x<0$. Since $f'(x)$ changes from negative to positive when passing through $x=0$, the first derivative test tells us there is a local minimum at $x=0$ which is $f(0)=0$. Further, for $x>0$, $f(x)>f(0)=0$ gives

$$\log(1+x) - \frac{x}{1+x} > 0 \Rightarrow \log(1+x) > \frac{x}{1+x} \quad \text{on } (0,\infty).$$ ■

Example 7.54

Find the intervals of monotonicity and local extrema of the function $f(x) = x\log x + 3x$.

Solution

The given function is defined and is differentiable at all $x \in (0,\infty)$.

$$f(x) = x\log x + 3x.$$

$$\text{Therefore} \quad f'(x) = \log x + 1 + 3 = 4 + \log x.$$

The stationary numbers are given by $4+\log x = 0$.

That is $x = e^{-4}$.

Hence the intervals of monotonicity are $(0, e^{-4})$ and (e^{-4}, ∞).

At $x = e^{-5} \in (0, e^{-4}), f'(e^{-5}) = -1 < 0$ and hence in the interval $(0, e^{-4})$ the function is strictly decreasing.

At $x = e^{-3} \in (e^{-4}, \infty), f'(e^{-3}) = 1 > 0$ and hence strictly increasing in the interval (e^{-4}, ∞). Since $f'(x)$ changes from negative to positive when passing through $x = e^{-4}$, the first derivative test tells us there is a local minimum at $x = e^{-4}$ and it is $f(e^{-4}) = -e^{-4}$. ■

Example 7.55

Find the intervals of monotonicity and local extrema of the function $f(x) = \frac{1}{1+x^2}$.

Solution

The given function is defined and is differentiable at all $x \in (-\infty, \infty)$. As

$$f(x) = \frac{1}{1+x^2},$$

$$\text{we have} \quad f'(x) = -\frac{2x}{(1+x^2)^2}.$$

The stationary numbers are given by $-\dfrac{2x}{(1+x^2)^2} = 0$ that is $x=0$.

Hence the intervals of monotonicity are $(-\infty,0)$ and $(0,\infty)$.

On the interval $(-\infty,0)$ the function strictly increases because $f'(x)>0$ in that interval.

The function $f(x)$ strictly decreases in the interval $(0,\infty)$ because $f'(x)<0$ in that interval. Since $f'(x)$ changes from positive to negative when passing through $x=0$, the first derivative test tells us there is local maximum at $x=0$ and the local maximum value is $f(0)=1$. ■

Example 7.56

Find the intervals of monotonicity and local extrema of the function $f(x)=\dfrac{x}{1+x^2}$.

Solution

The given function is defined and differentiable at all $x\in(-\infty,\infty)$, As

$$f(x) = \frac{x}{1+x^2},$$

$$f'(x) = \frac{1-x^2}{(1+x^2)^2}.$$

The stationary numbers are given by $1-x^2 = 0$ that is $x=\pm 1$.

Hence the intervals of monotonicity are $(-\infty,-1),(-1,1)$ and $(1,\infty)$.

Interval	(-∞, -1)	(-1, 1)	(1, ∞)
Sign of $f'(x)$	−	+	−
Monotonicity	strictly decreasing	strictly increasing	strictly decreasing

Table 7.6

Therefore, $f(x)$ strictly increasing on $(-\infty,-1)$ and $(1,\infty)$, strictly decreasing on $(-1,1)$.

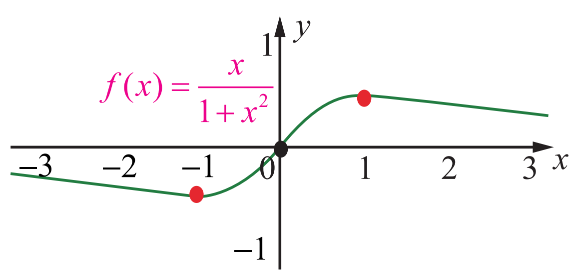


Fig.7.22

Since $f'(x)$ changes from negative to positive when passing through $x=-1$, the first derivative test tells us there is a local minimum at $x=-1$ and the local minimum value is $f(-1)=-\dfrac{1}{2}$. As $f'(x)$ changes from positive to negative when passing through $x=1$, the first derivative test tells us there is a local maximum at $x=1$ and the local maximum value is $f(1)=\dfrac{1}{2}$. ■

EXERCISE 7.6

1. Find the absolute extrema of the following functions on the given closed interval.

   (i) $f(x) = x^2 - 12x + 10$; $[1, 2]$ (ii) $f(x) = 3x^4 - 4x^3$; $[-1, 2]$

   (iii) $f(x) = 6x^{\frac{4}{3}} - 3x^{\frac{1}{3}}$; $[-1, 1]$ (iv) $f(x) = 2\cos x + \sin 2x$; $\left[0, \frac{\pi}{2}\right]$

2. Find the intervals of monotonicities and hence find the local extremum for the following functions:

   (i) $f(x) = 2x^3 + 3x^2 - 12x$ (ii) $f(x) = \frac{x}{x-5}$

   (iii) $f(x) = \frac{e^x}{1-e^x}$ (iv) $f(x) = \frac{x^3}{3} - \log x$

   (v) $f(x) = \sin x \cos x + 5, \ x \in (0, 2\pi)$

D8Q8X4

7.7 Applications of Second Derivative

Second derivative of a function is used to determine the concavity, convexity, the points of inflection, and local extrema of functions.

7.7.1 Concavity, Convexity, and Points of Inflection

A graph is said to be concave down (convex up) at a point if the tangent line lies above the graph in the vicinity of the point. It is said to be concave up (convex down) at a point if the tangent line to the graph at that point lies below the graph in the vicinity of the point. This may be easily observed from the adjoining graph.

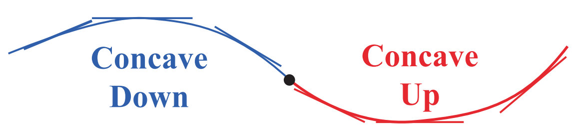


Fig.7.23

Definition 7.8

Let $f(x)$ be a function whose second derivative exists in an open interval $I = (a, b)$. Then the function $f(x)$ is said to be

(i) If $f'(x)$ is strictly increasing on I, then the function is concave up on an open interval I.

(ii) If $f'(x)$ is strictly decreasing on I, then the function is concave down on an open interval I.

Analytically, given a differentiable function whose graph $y = f(x)$, then the concavity is given by the following result.

Theorem 7.11 (Test of Concavity)

(i) If $f''(x) > 0$ on an open interval I, then $f(x)$ is concave up on I.

(ii) If $f''(x) < 0$ on an open interval I, then $f(x)$ is concave down on I.

Remark

(1) Any local maximum of a convex upward function defined on the interval $[a, b]$ is also its absolute maximum on this interval.

(2) Any local minimum of a convex downward function defined on the interval $[a, b]$ is also its absolute minimum on this interval.

(3) There is only one absolute maximum (and one absolute minimum) but there can be more than one local maximum or minimum.

Points of Inflection

Definition 7.9

The points where the graph of the function changes from **"concave up to concave down"** or **"concave down to concave up"** are called the **points of inflection of** $f(x)$.

Theorem 7.12 (Test for Points of Inflection)

(i) If $f''(c)$ exists and $f''(c)$ changes sign when passing through $x = c$, then the point $(c, f(c))$ is a point of inflection of the graph of f.

(ii) If $f''(c)$ exists at the point of inflection, then $f''(c) = 0$.

Remark

To determine the position of points of inflexion on the curve $y = f(x)$ it is necessary to find the points where $f''(x)$ changes sign. For 'smooth' curves (no sharp corners), this may happen when either

(i) $f''(x) = 0$ or

(ii) $f''(x)$ does not exist at the point.

Remark

(1) It is also possible that $f''(c)$ may not exist, but $(c, f(c))$ could be a point of inflection. For instance, $f(x) = x^{\frac{1}{3}}$ at $c = 0$.

(2) It is possible that $f''(c) = 0$ at a point but $(c, f(c))$ need not be a point of inflection. For instance, $f(x) = x^4$ at $c = 0$.

(3) A point of inflection need not be a stationary point. For instance, if $f(x) = \sin x$ then, $f'(x) = \cos x$ and $f''(x) = -\sin x$ and hence $(\pi, 0)$ is a point of inflection but not a stationary point for $f(x)$.

Example 7.57

Determine the intervals of concavity of the curve $f(x) = (x-1)^3 \cdot (x-5), x \in \mathbb{R}$ and, points of inflection if any.

Solution

The given function is a polynomial of degree 4. Now,

$$
\begin{aligned}
f'(x) &= (x-1)^3 + 3(x-1)^2 \cdot (x-5) \\
&= 4(x-1)^2 \cdot (x-4) \\
f''(x) &= 4((x-1)^2 + 2(x-1) \cdot (x-4)) \\
&= 12(x-1) \cdot (x-3)
\end{aligned}
$$

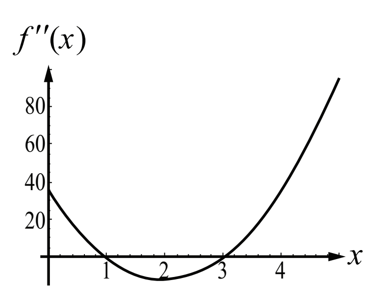


Fig.7.24

Now,

$$f''(x) = 0 \quad \Rightarrow x = 1, x = 3.$$

The intervals of concavity are tabulated in Table 7.7.

Interval	(-∞, 1)	(1, 3)	(3, ∞)
Sign of $f''(x)$	+	–	+
Concavity	concave up	concave down	concave up

Table 7.7

The curve is concave upwards on $(-\infty, 1)$ and $(3, \infty)$.

The curve is concave downwards on $(1, 3)$.

As $f''(x)$ changes its sign when it passes through $x = 1$ and $x = 3$, $(1, f(1)) = (1, 0)$ and $(3, f(3)) = (3, -16)$ are points of inflection for the graph $y = f(x)$. The sign change may be observed from the adjoining figure of the curve $f''(x)$. ■

Example 7.58

Determine the intervals of concavity of the curve $y = 3 + \sin x$.

Solution

The given function is a periodic function with period 2π and hence there will be stationary points and points of inflections in each period interval. We have,

$$\frac{dy}{dx} = \cos x \text{ and } \frac{d^2y}{dx^2} = -\sin x$$

$$\text{Now, } \frac{d^2y}{dx^2} = -\sin x = 0 \Rightarrow x = n\pi .$$

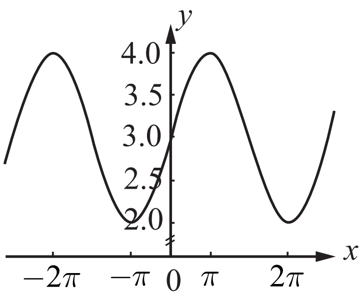


Fig.7.25

We now consider an interval, $(-\pi, \pi)$ by splitting into two sub intervals $(-\pi, 0)$ and $(0, \pi)$.

In the interval $(-\pi, 0)$, $\frac{d^2y}{dx^2} > 0$ and hence the function is concave upward.

In the interval $(0, \pi)$, $\frac{d^2y}{dx^2} < 0$ and hence the function is concave downward. Therefore $(0, 3)$ is a point of inflection (see Fig. 7.25). The general intervals need to be considered to discuss the concavity of the curve are $(n\pi, (n+1)\pi)$, where n is any integer which can be discussed as before to conclude that $(n\pi, 3)$ are also points of inflection. ■

7.7.2 Extrema using Second Derivative Test

The Second Derivative Test: The Second Derivative Test relates the concepts of critical points, extreme values, and concavity to give a very useful tool for determining whether a critical point on the graph of a function is a relative minimum or maximum.

Theorem 7.13 (The Second Derivative Test)

Suppose that c is a critical point at which $f'(c) = 0$, that $f'(x)$ exists in a neighborhood of c, and that $f''(c)$ exists. Then f has a relative maximum value at c if $f''(c) < 0$ and a relative minimum value at c if $f''(c) > 0$. If $f''(c) = 0$, the test is not informative.

Example 7.59

Find the local extremum of the function $f(x)=x^4+32x$.

Solution

We have,

$$f'(x) = 4x^3+32=0 \text{ gives } x^3=-8$$

$$\Rightarrow x = -2$$

$$\text{and } f''(x) = 12x^2.$$

As $f''(-2)>0$, the function has local minimum at $x=-2$. The local minimum value is $f(-2)=-48$. Therefore, the extreme point is $(-2,-48)$. ■

Example 7.60

Find the local extrema of the function $f(x)=4x^6-6x^4$.

Solution

Differentiating with respect to x, we get

$$\begin{aligned} f'(x) &= 24x^5-24x^3 \\ &= 24x^3(x^2-1) \\ &= 24x^3(x+1)(x-1) \end{aligned}$$

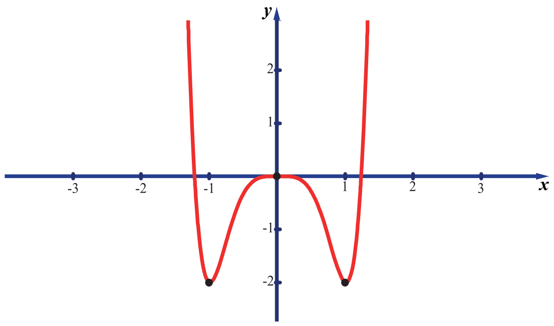


Fig.7.26

$f'(x)=0 \Rightarrow x=-1,\ 0,\ 1$. Hence the critical numbers are $x=-1,0,1$.

Now, $f''(x)=120x^4-72x^2=24x^2(5x^2-3)$.

$\Rightarrow f''(-1)=48,\ f''(0)=0,\ f''(1)=48$.

As $f''(-1)$ and $f''(1)$ are positive by the second derivative test, the function $f(x)$ has local minimum. But at $x=0$, $f''(0)=0$. That is the second derivative test does not give any information about local extrema at $x=0$. Therefore, we need to go back to the first derivative test. The intervals of monotonicity is tabulated in Table 7.8.

Interval	$(-\infty,-1)$	$(-1,0)$	$(0,1)$	$(1,\infty)$
Sign of $f'(x)$	$-$	$+$	$-$	$+$
Monotonicity	strictly decreasing	strictly increasing	strictly decreasing	strictly increasing

Table 7.8

By the first derivative test $f(x)$ has local minimum at $x=-1$, its local minimum value is -2. At $x=0$, the function $f(x)$ has local maximum at $x=0$, and its local maximum value is 0. At $x=1$, the function $f(x)$ has local minimum at $x=1$, and its local minimum value is -2. ■

Remark

When the second derivative vanishes, we have no information about extrema. We have used the first derivative test to find out the extrema of the function!

Example 7.61

Find the local maximum and minimum of the function x^2y^2 on the line $x + y = 10$.

Solution

Let the given function be written as $f(x) = x^2(10 - x)^2$. Now,

$$\begin{aligned} f(x) &= x^2(100 - 20x + x^2) = x^4 - 20x^3 + 100x^2 \\ \text{Therefore,}\quad f'(x) &= 4x^3 - 60x^2 + 200x = 4x(x^2 - 15x + 50) \\ f'(x) &= 4x(x^2 - 15x + 50) = 0 \Rightarrow x = 0,\ 5,\ 10 \\ \text{and}\quad f''(x) &= 12x^2 - 120x + 200 \end{aligned}$$

The stationary numbers of $f(x)$ are $x = 0, 5, 10$ at these points the values of $f''(x)$ are respectively $200, -100$ and 200. At $x = 0$, it has local minimum and its value is $f(0) = 0$. At $x = 5$, it has local maximum and its value is $f(5) = 625$. At $x = 10$, it has local minimum and its value is $f(10) = 0$. ■

EXERCISE 7.7

1. Find intervals of concavity and points of inflexion for the following functions:
   (i) $f(x) = x(x-4)^3$ (ii) $f(x) = \sin x + \cos x,\ 0 < x < 2\pi$ (iii) $f(x) = \frac{1}{2}(e^x - e^{-x})$
2. Find the local extrema for the following functions using second derivative test :
   (i) $f(x) = -3x^5 + 5x^3$ (ii) $f(x) = x \log x$ (iii) $f(x) = x^2 e^{-2x}$
3. For the function $f(x) = 4x^3 + 3x^2 - 6x + 1$ find the intervals of monotonicity, local extrema, intervals of concavity and points of inflection.

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7.8 Applications in Optimization

Optimization is a process of finding an extreme value (either maximum or minimum) under certain conditions.

A procedure for solving for an extremum or optimization problems.

Step 1 : Draw an appropriate figure and label the quantities relevant to the problem.

Step 2 : Find a experssion for the quantity to be maximized or minimized.

Step 3 : Using the given conditions of the problem, the quantity to be extremized .

Step 4 : Determine the interval of possible values for this variable from the conditions given in the problem.

Step 5 : Using the techniques of extremum (absolute extrimum, first derivative test or second derivative test) obtain the maximum or minimum.

Example 7.62

We have a 12 unit square piece of thin material and want to make an open box by cutting small squares from the corners of our material and folding the sides up. The question is, which cut produces the box of maximum volume?

Solution

Let x = length of the cut on each side of the little squares.

V = the volume of the folded box.

The length of the base after two cuts along each edge of size x is $12-2x$. The depth of the box after folding is x, so the volume is $V = x \times (12-2x)^2$. Note that, when $x=0$ or 6, the volume is zero and hence there cannot be a box. Therefore the problem is to maximize, $V = x \times (12-2x)^2, x \in (0,6)$.

Now,
$$\frac{dV}{dx} = (12-2x)^2 - 4x(12-2x)$$
$$= (12-2x)(12-6x).$$

$\frac{dV}{dx} = 0$ gives the stationary numbers $x = 2, 6$. Since $6 \notin (0,6)$ the only stationary number is at $x = 2 \in (0,6)$. Further, $\frac{dV}{dx}$ changes its sign from postive to negative when passing through $x=2$. Therefore at $x=2$ the volume V is local maximum. The local maximum volume value is $V = 128$ units. Hence the maximum cut can only be 2 units.

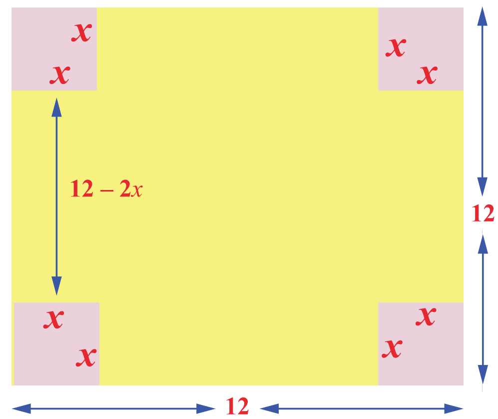

Fig.7.27

Example 7.63

Find the points on the unit circle $x^2 + y^2 = 1$ nearest and farthest from $(1,1)$.

Solution

The distance from the point $(1,1)$ to any point (x,y) is $d = \sqrt{(x-1)^2 + (y-1)^2}$. Instead of extremising d, for convenience we extremise $D = d^2 = (x-1)^2 + (y-1)^2$, subject to the condition $x^2 + y^2 = 1$. Now, $\frac{dD}{dx} = 2(x-1) + 2(y-1) \times \frac{dy}{dx}$, where the $\frac{dy}{dx}$ will be computed by differentiating $x^2 + y^2 = 1$ with respect to x. Therefore, we get $2x + 2y\frac{dy}{dx} = 0$ which gives us $\frac{dy}{dx} = -\frac{x}{y}$.

Substituting this, we get
$$\frac{dD}{dx} = 2(x-1) + 2(y-1)\left(-\frac{x}{y}\right)$$
$$= \frac{2[xy - y - xy + x]}{y}$$
$$\frac{dD}{dx} = 2\left[\frac{x-y}{y}\right] = 0$$
$$\Rightarrow x = y$$

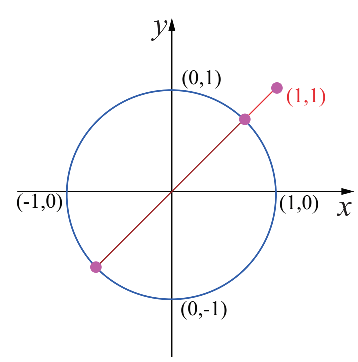

Fig.7.28

Since (x,y) lie on the circle $x^2 + y^2 = 1$, we get $2x^2 = 1$ which gives $x = \pm\frac{1}{\sqrt{2}}$. Hence the points at which the extremum distance occur are, $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

To find the extrema, we apply second derivative test. So,

$$\frac{d^2D}{dx^2} = 2\frac{y^2+x^2}{y^3}.$$

The value of $\left(\frac{d^2D}{dx^2}\right)_{\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)} > 0$; $\left(\frac{d^2D}{dx^2}\right)_{\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)} < 0$.

This implies the nearest and farthest points are $\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$ respectively.

Therefore, the nearest and the farthest distances are respectively $\sqrt{2}-1$ and $\sqrt{2}+1$. ■

Example 7.64

A steel plant is capable of producing x tonnes per day of a low-grade steel and y tonnes per day of a high-grade steel, where $y = \frac{40-5x}{10-x}$. If the fixed market price of low-grade steel is half that of high-grade steel, then what should be optimal productions in low-grade steel and high-grade steel in order to have maximum receipts.

Solution

Let the price of low-grade steel be ₹p per tonne. Then the price of high-grade steel is ₹$2p$ per tonne.

The total receipt per day is given by $R = px + 2py = px + 2p\left(\frac{40-5x}{10-x}\right)$. Hence the problem is to maximise R. Now, simplifying and differentiating R with respect to x, we get

$$R = p\left(\frac{80-x^2}{10-x}\right)$$

$$\frac{dR}{dx} = p\left(\frac{x^2-20x+80}{(10-x)^2}\right)$$

$$\frac{d^2R}{dx^2} = -\frac{40p}{(10-x)^3}$$

Now, $\frac{dR}{dx} = 0 \Rightarrow x^2-20x+80=0$ and hence $x = 10 \pm 2\sqrt{5}$

At $x = 10-2\sqrt{5}$, $\frac{d^2R}{dx^2} < 0$

and hence R will be maximum. If $x = 10-2\sqrt{5}$ then $y = 5-\sqrt{5}$. Therefore the steel plant must produce low-grade and high-grade steels respectively in tonnes per day are

$$10-2\sqrt{5} \text{ and } 5-\sqrt{5}.$$ ■

Example 7.65

Prove that among all the rectangles of the given area square has the least perimeter.

Solution

Let x, y be the sides of the rectangle. Hence the area of the rectangle is $xy = k$ (given). The perimeter of the rectangle P is $2(x+y)$. So the problem is to minimize $2(x+y)$ suject to the condition $xy = k$. Let $P(x) = 2\left(x+\frac{k}{x}\right)$.

$$P'(x) = 2\left(1 - \frac{k}{x^2}\right)$$

$$P'(x) = 0 \Rightarrow x = \pm\sqrt{k}.$$

As x, y are sides of the rectangle, $x = \sqrt{k}$ is a critical number.

Now, $P''(x) = \frac{4k}{x^3}$ and $P''(\sqrt{k}) > 0 \Rightarrow P(x)$ has its minimum value at $x = \sqrt{k}$.

Substituting $x = \sqrt{k}$ in $xy = k$ we get $y = \sqrt{k}$. Therefore the minimum perimeter rectangle of a given area is a square. ■

EXERCISE 7.8

1. Find two positive numbers whose sum is 12 and their product is maximum.
2. Find two positive numbers whose product is 20 and their sum is minimum.
3. Find the smallest possible value of $x^2 + y^2$ given that $x + y = 10$.
4. A garden is to be laid out in a rectangular area and protected by wire fence. What is the largest possible area of the fenced garden with 40 metres of wire.
5. A rectangular page is to contain 24 cm^2 of print. The margins at the top and bottom of the page are 1.5 cm and the margins at other sides of the page is 1 cm. What should be the dimensions of the page so that the area of the paper used is minimum.
6. A farmer plans to fence a rectangular pasture adjacent to a river. The pasture must contain 1,80,000 sq.mtrs in order to provide enough grass for herds. No fencing is needed along the river. What is the length of the minimum needed fencing material?
7. Find the dimensions of the rectangle with maximum area that can be inscribed in a circle of radius 10 cm.
8. Prove that among all the rectangles of the given perimeter, the square has the maximum area.
9. Find the dimensions of the largest rectangle that can be inscribed in a semi circle of radius r cm.
10. A manufacturer wants to design an open box having a square base and a surface area of 108 sq.cm. Determine the dimensions of the box for the maximum volume.
11. The volume of a cylinder is given by the formula $V = \pi r^2 h$. Find the greatest and least values of V if $r + h = 6$.
12. A hollow cone with base radius a cm and height b cm is placed on a table. Show that the volume of the largest cylinder that can be hidden underneath is $\frac{4}{9}$ times volume of the cone.

7.9 Symmetry and Asymptotes

7.9.1 Symmetry

Consider the following curves and observe that each of them is having some special properties, called symmetry with respect to a point, with respect to a line.

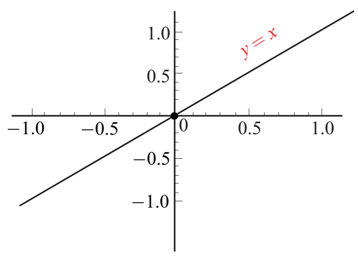

Fig.7.29

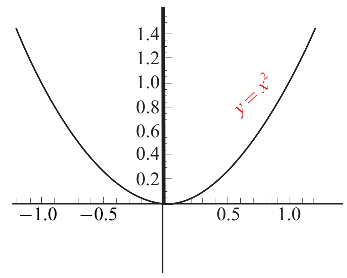

Fig.7.30

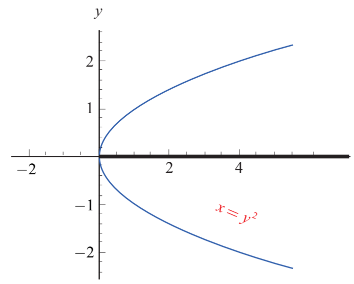

Fig. 7.31

We now formally define the symmetry as follows :

If an image or a curve is a mirror reflection of another image with respect to a line, we say the image or the curve is symmetric with respect to that line. The line is called the line of symmetry.

A curve is said to have a θ angle rotational symmetry with respect to a point if the curve is unchanged by a rotation of an angle θ with respect to that point.

A curve may be symmetric with respect to many lines. Specifically, we consider the symmetry with respect to the co-ordinate axes and symmetric with respect to the origin. Mathematically, a curve $f(x, y) = 0$ is said to be

- **Symmetric with respect to the *y*-axis** if $f(x, y) = f(-x, y)$ $\forall x, y$ or if (x, y) is a point on the graph of the curve then so is $(-x, y)$. If we keep a mirror on the *y*-axis the portion of the curve on one side of the mirror is the same as the portion of the curve on the other side of the mirror.
- **Symmetric with respect to the *x*-axis** if $f(x, y) = f(x, -y)$ $\forall x, y$ or if (x, y) is a point on the graph of the curve then so is $(x, -y)$. If we keep a mirror on the x-axis the portion of the curve on one side of the mirror is the same as the portion of the curve on the other side of the mirror.
- **Symmetric with respect to the origin** if $f(x, y) = f(-x, -y)$ $\forall x, y$ or if (x, y) is a point on the graph of the curve then so is $(-x, -y)$. That is the curve is unchanged if we rotate it by $180°$ about the origin.

For instance, the curves mentioned above $x = y^2, y = x^2$ and $y = x$ are symmetric with respect to *x*-axis, *y*-axis and origin respectively.

7.9.2 Asymptotes

An asymptote for the curve $y = f(x)$ is a straight line which is a tangent at ∞ to the curve. In other words the distance between the curve and the straight line tends to zero when the points on the curve approach infinity. There are three types of asymptotes. They are

1. **Horizontal asymptote**, which is parallel to the x-axis. The line $y = L$ is said to be a horizontal asymptote for the curve $y = f(x)$ if either $\lim\limits_{x \to +\infty} f(x) = L$ or $\lim\limits_{x \to -\infty} f(x) = L$.
2. **Vertical asymptote**, which is parallel to the y-axis. The line $x = a$ is said to be vertical asymptote for the curve $y = f(x)$ if $\lim\limits_{x \to a^-} f(x) = \pm\infty$ or $\lim\limits_{x \to a^+} f(x) = \pm\infty$.
3. **Slant asymptote**, A slant (oblique) asymptote occurs when the polynomial in the numerator is a higher degree than the polynomial in the denominator.

   To find the slant asymptote you must divide the numerator by the denominator using either long division or synthetic division.

Example 7.66

Find the asymptotes of the function $f(x)=\dfrac{1}{x}$.

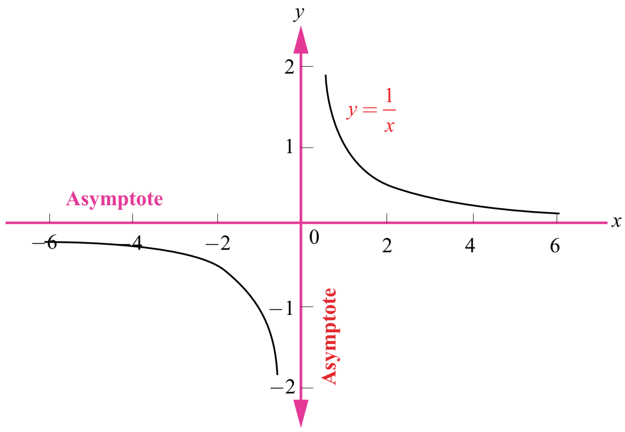

Fig.7.32

Solution

We have, $\lim\limits_{x\to 0^-}\dfrac{1}{x}=-\infty$ and $\lim\limits_{x\to 0^+}\dfrac{1}{x}=\infty$. Hence, the required vertical asymptote is $x=0$ or the y-axis.

As the curve is symmetric with respect to both the axes, $y=0$ or the x-axis is also an asymptote. Hence this (rectangular hyperbola) curve has both the vertical and horizontal asymptotes. ■

Example 7.67

Find the slant (oblique) asymptote for the function $f(x)=\dfrac{x^2-6x+7}{x+5}$.

Solution

Since the polynomial in the numerator is a higher degree (2nd) than the denominator (1st), we know we have a slant asymptote. To find it, we must divide the numerator by the denominator. We can use long division to do that:

$$\begin{array}{r|l} & x-11 \\ \hline x+5 & x^2-6x+7 \\ & x^2+5x \\ \hline & -11x+7 \\ & -11x-55 \\ \hline & 62 \end{array}$$

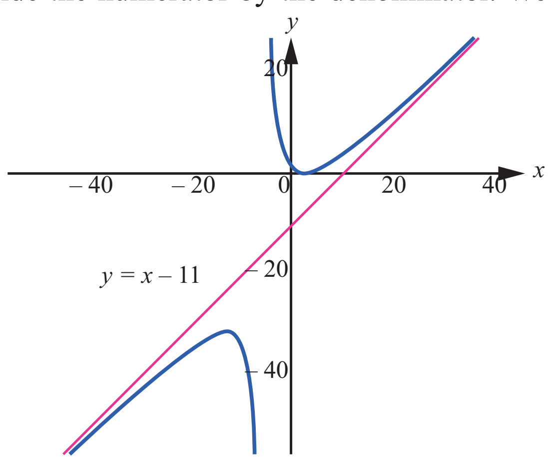

Fig.7.33

Notice that we don't need to finish the long division problem to find the remainder. We only need the terms that will make up the equation of the line. The slant asymptote is $y=x-11$. ■

As you can see in this graph of the function, the curve approaches the slant asymptote $y=x-11$ but never crosses it:

Example 7.68

Find the asymptotes of the curve $f(x)=\dfrac{2x^2-8}{x^2-16}$.

Solution

As $\lim\limits_{x\to -4^+}\dfrac{2x^2-8}{x^2-16}=-\infty$ and $\lim\limits_{x\to 4^+}\dfrac{2x^2-8}{x^2-16}=\infty$.

Therefore $x=-4$ and $x=4$ are vertical asymptotes.

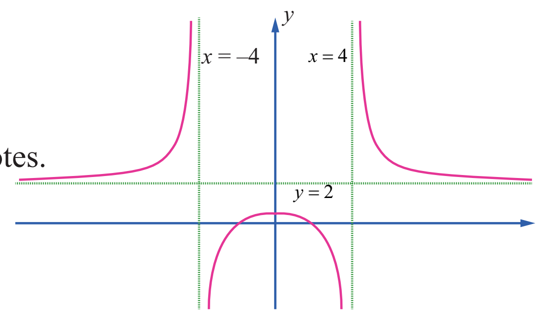

Fig.7.34

As $\lim\limits_{x\to\infty}\dfrac{2x^2-8}{x^2-16}=\lim\limits_{x\to\infty}\dfrac{2-\dfrac{8}{x^2}}{1-\dfrac{16}{x^2}}=2$

$$\text{and} \quad \lim_{x\to-\infty} \frac{2x^2-8}{x^2-16} = \lim_{x\to-\infty} \frac{2-\frac{8}{x^2}}{1-\frac{16}{x^2}} = 2$$

Therefore, $y = 2$ is a horizontal asymptote. This can also be obtained by synthetic division. ■

7.10 Sketching of Curves

When we are sketching the graph of functions either by hand or through any graphing software we cannot show the entire graph. Only a part of the graph can be sketched. Hence a crucial question is which part of the curve we need to show and how to decide that part. To decide on this we use the derivatives of functions. We enlist few guidelines for determining a good viewing rectangle for the graph of a function. They are :

(i) The domain and the range of the function.

(ii) The intercepts of the cure (if any).

(iii) Critical points of the function.

(iv) Local extrema of the function.

(v) Intervals of concavity.

(vi) Points of inflexions (if any).

(vii) Asymptotes of the curve (if exists)

Example 7.69

Sketch the curve $y = f(x) = x^2 - x - 6$.

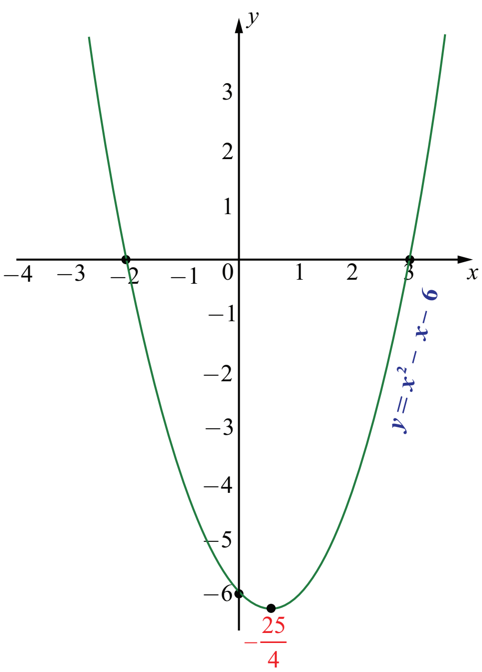

Fig.7.35

Solution

Factorising the given function, we have

$y = f(x) = (x-3)(x+2)$.

(1) The domain of the given function $f(x)$ is the entire real line.

(2) Putting $y = 0$ we get $x = -2, 3$. Therefore the x-intercepts are $(-2,0)$ and $(3,0)$ putting $x = 0$ we get $y = -6$. Therefore the y-intercept is $(0,-6)$.

(3) $f'(x) = 2x - 1$ and hence the critical point of the curve occurs at $x = \frac{1}{2}$.

(4) $f''(x) = 2 > 0, \forall x$. Therefore at $x = \frac{1}{2}$ the curve has a local minimum which is $f\left(\frac{1}{2}\right) = -\frac{25}{4}$.

(5) The range of the function is $y \geq -\frac{25}{4}$

(6) Since $f''(x) = 2 > 0, \forall x$ the function is concave upward in the entire real line.

(7) Since $f(x) = 2 \neq 0, \forall x$ the curve has no points of inflection.

(8) The curve has no asymptotes.

The rough sketch of the curve is shown on the right side. ■

Example 7.70

Sketch the curve $y = f(x) = x^3 - 6x - 9$.

Solution

Factorising the given function, we have

$$y = f(x) = (x-3)(x^2+3x+3).$$

(1) The domain and the range of the given function $f(x)$ are the entire real line.

(2) Putting $y = 0$, we get the $x = 3$. The other two roots are imaginary. Therefore, the x-intercept is $(3,0)$. Putting $x = 0$, we get $y = -9$. Therefore, the y-intercept is $(0,-9)$.

(3) $f'(x) = 3(x^2-2)$ and hence the critical points of the curve occur at $x = \pm\sqrt{2}$.

(4) $f''(x) = 6x$. Therefore at $x = \sqrt{2}$ the curve has a local minimum because $f''\left(\sqrt{2}\right) = 6\sqrt{2} > 0$. The local minimum is $f\left(\sqrt{2}\right) = -4\sqrt{2} - 9$. Similarly $x = -\sqrt{2}$ the curve has a local maximum because $f''\left(-\sqrt{2}\right) = -6\sqrt{2} < 0$. The local maximum is $f\left(-\sqrt{2}\right) = 4\sqrt{2} - 9$.

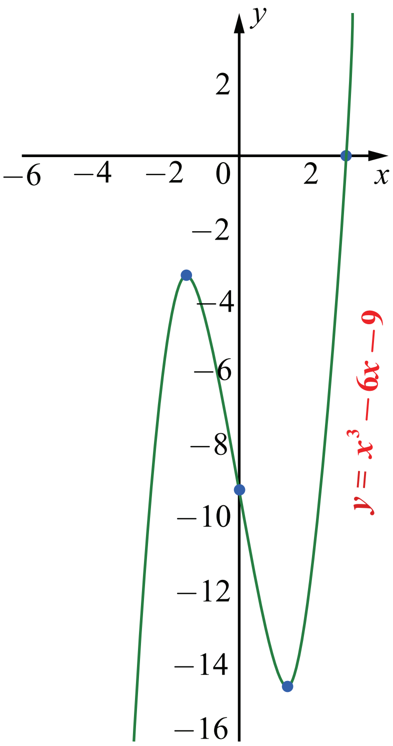

Fig.7.36

(5) Since $f''(x) = 6x > 0,\ \forall\, x > 0$ the function is concave upward in the positive real line. As $f''(x) = 6x < 0,\ \forall\, x < 0$ the function is concave downward in the negative real line.

(6) Since $f''(x) = 0$ at $x = 0$ and $f''(x)$ changes its sign when passing through $x = 0$. Therefore the point of inflection is $(0, f(0)) = (0,-9)$.

(7) The curve has no asymptotes.

The rough sketch of the curve is shown on the right side. ∎

Example 7.71

Sketch the curve $y = \dfrac{x^2 - 3x}{(x-1)}$.

Solution

Factorising the given function we have,

$$y = f(x) = \frac{x(x-3)}{(x-1)}.$$

(1) The domain and the range of $f(x)$ are respectively $R \setminus \{1\}$ and the entire real line.

(2) Putting $y = 0$ we get the $x = 0, 3$. Therefore the x-intercept is $(3,0)$. Putting $x = 0$, we get $y = 0$. Therefore the curve passes through the origin.

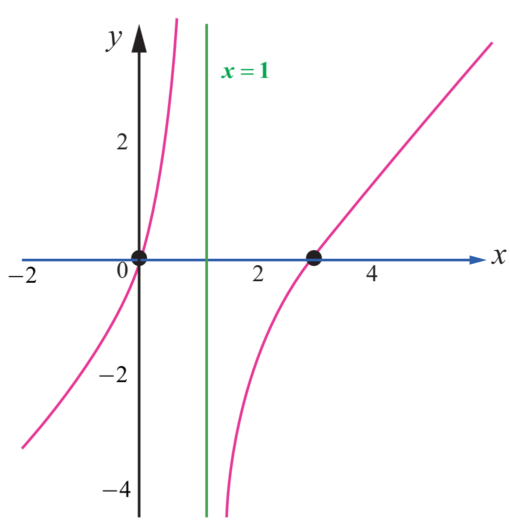

Fig.7.37

(3) $f'(x) = \frac{x^2 - 2x + 3}{(x-1)^2}$ and hence the critical point of the curve occurs at $x = 1$ as $f'(1)$ does not exist. But $x^2 - 2x + 3 = 0$ has no real solution. Hence the only critical point occurs at $x = 1$.

(4) $x = 1$ is not in the domain of the function and $f'(x) \neq 0 \; \forall \; x \in \mathbb{R} \setminus \{1\}$, there is no local maximum or local minimum.

(5) $f''(x) = -\frac{4}{(x-1)^3} \; \forall x \in \mathbb{R} \setminus \{1\}$. Therefore when $x < 1, f''(x) > 0$ the curve is concave upwards in $(-\infty, 1)$ and when $x > 1, f''(x) < 0$ the curve is concave downwards in $(1, \infty)$. Since $x = 1$ is not in the domain $f''(x) \neq 0 \; x \in \mathbb{R} \setminus \{1\}$ there is no point of infection for $f(x)$.

(6) Since, $\lim\limits_{x \to 1^-} \frac{x^2 - 3x}{(x-1)} = +\infty$ and $\lim\limits_{x \to 1^+} \frac{x^2 - 3x}{(x-1)} = -\infty, x = 1$ is a vertical asymptote.

The rough sketch is shown on the right side.

Example 7.72

Sketch the graph of the function $y = \frac{3x}{x^2 - 1}$.

Solution

(1) The domain of $f(x)$ is $\mathbb{R} \setminus \{-1, 1\}$.

(2) Since $f(-x, -y) = f(x, y)$, the curve is symmetric about the origin.

(3) Putting $y = 0$, we get $x = 0$. Hence the x-intercept is $(0, 0)$.

(4) Putting $x = 0$, we get $y = 0$. Hence the y-intercept is $(0, 0)$.

(5) To determine monotonicity, we find the first derivative as $f'(x) = \frac{-3(x^2 + 1)}{(x^2 - 1)^2}$.

Hence, $f'(x)$ does not exist at $x = -1, 1$. Therefore, critical numbers are $x = -1, 1$. The intervals of monotonicity is tabulated in Table 7.9.

Interval	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
Sign of $f'(x)$	$-$	$-$	$-$
Monotonicity	strictly decreasing	strictly decreasing	strictly decreasing

Table 7.9

(6) Since there is no sign change in $f'(x)$ when passing through critical numbers. There is no local extrema.

(7) To determine the concavity, we find the second derivative as $f''(x) = \frac{6x(x^2 + 3)}{(x^2 - 1)^3}$.

$f''(x) = 0 \Rightarrow x = 0$ and $f''(x)$ does not exist at $x = -1, 1$.

The intervals of concavity is tabulated in Table 7.10.

Interval	(-∞, -1)	(-1, 0)	(0, 1)	(1, ∞)
Sign of $f''(x)$	$-$	$+$	$-$	$+$
Concavity	concave down	concave up	concave down	concave up

Table 7.10

(8) As $x=-1$ and 1 are not in the domain of $f(x)$ and at $x=0$, the second derivative is zero and $f''(x)$ changes its sign from positive to negative when passing through $x=0$. .Therefore, the point of inflection is $(0, f(0))=(0,0)$.

(9) $\lim\limits_{x\to\pm\infty} f(x)=\lim\limits_{x\to\pm\infty}\dfrac{3x}{x^2-1}=\lim\limits_{x\to\pm\infty}\dfrac{3}{x-\dfrac{1}{x}}=0$. Therefore $y=0$ is a horizontal asymptote. Since the denominator is zero, when $x=\pm 1$.

$\lim\limits_{x\to -1^-}\dfrac{3x}{x^2-1}=-\infty$, $\lim\limits_{x\to -1^+}\dfrac{3x}{x^2-1}=+\infty$,

$\lim\limits_{x\to 1^-}\dfrac{3x}{x^2-1}=-\infty$, $\lim\limits_{x\to 1^+}\dfrac{3x}{x^2-1}=\infty$.

Therefore $x=-1$ and $x=1$ are vertical asymptotes.

The rough sketch of the curve is shown on the right side.

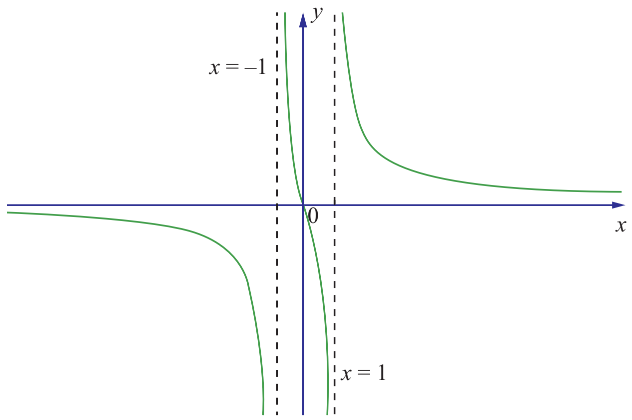

Fig.7.38

EXERCISE 7.9

1. Find the asymptotes of the following curves :

(i) $f(x)=\dfrac{x^2}{x^2-1}$ (ii) $f(x)=\dfrac{x^2}{x+1}$ (iii) $f(x)=\dfrac{3x}{\sqrt{x^2+2}}$

(iv) $f(x)=\dfrac{x^2-6x-1}{x+3}$ (v) $f(x)=\dfrac{x^2+6x-4}{3x-6}$

2. Sketch the graphs of the following functions:

(i) $y=-\dfrac{1}{3}(x^3-3x+2)$ (ii) $y=x\sqrt{4-x}$ (iii) $y=\dfrac{x^2+1}{x^2-4}$

(iv) $y=\dfrac{1}{1+e^{-x}}$

EXERCISE 7.10

X6F3U3

Choose the correct or the most suitable answer from the given four alternatives :

1. The volume of a sphere is increasing in volume at the rate of $3\pi\,\text{cm}^3/\text{sec}$.

   The rate of change of its radius when radius is $\frac{1}{2}$ cm

   (1) 3 cm/s (2) 2 cm/s (3) 1 cm/s (4) $\frac{1}{2}$ cm/s

2. A balloon rises straight up at 10 m/s. An observer is 40 m away from the spot where the balloon left the ground. The rate of change of the balloon's angle of elevation in radian per second when the balloon is 30 metres above the ground.

   (1) $\frac{3}{25}$ radians/sec (2) $\frac{4}{25}$ radians/sec (3) $\frac{1}{5}$ radians/sec (4) $\frac{1}{3}$ radians/sec

3. The position of a particle moving along a horizontal line of any time t is given by $s(t) = 3t^2 - 2t - 8$. The time at which the particle is at rest is

   (1) $t = 0$ (2) $t = \frac{1}{3}$ (3) $t = 1$ (4) $t = 3$

4. A stone is thrown up vertically. The height it reaches at time t seconds is given by $x = 80t - 16t^2$. The stone reaches the maximum height in time t seconds is given by

   (1) 2 (2) 2.5 (3) 3 (4) 3.5

5. The point on the curve $6y = x^3 + 2$ at which y-coordinate changes 8 times as fast as x-coordinate is

   (1) $(4,11)$ (2) $(4,-11)$ (3) $(-4,11)$ (4) $(-4,-11)$

6. The abscissa of the point on the curve $f(x) = \sqrt{8-2x}$ at which the slope of the tangent is -0.25?

   (1) -8 (2) -4 (3) -2 (4) 0

7. The slope of the line normal to the curve $f(x) = 2\cos 4x$ at $x = \frac{\pi}{12}$ is

   (1) $-4\sqrt{3}$ (2) -4 (3) $\frac{\sqrt{3}}{12}$ (4) $4\sqrt{3}$

8. The tangent to the curve $y^2 - xy + 9 = 0$ is vertical when

   (1) $y = 0$ (2) $y = \pm\sqrt{3}$ (3) $y = \frac{1}{2}$ (4) $y = \pm 3$

9. Angle between $y^2 = x$ and $x^2 = y$ at the origin is

   (1) $\tan^{-1}\frac{3}{4}$ (2) $\tan^{-1}\left(\frac{4}{3}\right)$ (3) $\frac{\pi}{2}$ (4) $\frac{\pi}{4}$

10. The value of the limit $\lim\limits_{x\to 0}\left(\cot x - \frac{1}{x}\right)$ is

    (1) 0 (2) 1 (3) 2 (4) ∞

11. The function $\sin^4 x + \cos^4 x$ is increasing in the interval

(1) $\left[\frac{5\pi}{8}, \frac{3\pi}{4}\right]$ (2) $\left[\frac{\pi}{2}, \frac{5\pi}{8}\right]$ (3) $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$ (4) $\left[0, \frac{\pi}{4}\right]$

12. The number given by the Rolle's theorem for the function $x^3 - 3x^2, x \in [0,3]$ is

(1) 1 (2) $\sqrt{2}$ (3) $\frac{3}{2}$ (4) 2

13. The number given by the Mean value theorem for the function $\frac{1}{x}, x \in [1,9]$ is

(1) 2 (2) 2.5 (3) 3 (4) 3.5

14. The minimum value of the function $|3 - x| + 9$ is

(1) 0 (2) 3 (3) 6 (4) 9

15. The maximum slope of the tangent to the curve $y = e^x \sin x, x \in [0, 2\pi]$ is at

(1) $x = \frac{\pi}{4}$ (2) $x = \frac{\pi}{2}$ (3) $x = \pi$ (4) $x = \frac{3\pi}{2}$

16. The maximum value of the function $x^2 e^{-2x}, x > 0$ is

(1) $\frac{1}{e}$ (2) $\frac{1}{2e}$ (3) $\frac{1}{e^2}$ (4) $\frac{4}{e^4}$

17. One of the closest points on the curve $x^2 - y^2 = 4$ to the point $(6,0)$ is

(1) $(2,0)$ (2) $\left(\sqrt{5},1\right)$ (3) $\left(3,\sqrt{5}\right)$ (4) $\left(\sqrt{13},-\sqrt{3}\right)$

18. The maximum value of the product of two positive numbers, when their sum of the squares is 200, is

(1) 100 (2) $25\sqrt{7}$ (3) 28 (4) $24\sqrt{14}$

19. The curve $y = ax^4 + bx^2$ with $ab > 0$

(1) has no horizontal tangent (2) is concave up

(3) is concave down (4) has no points of inflection

20. The point of inflection of the curve $y = (x-1)^3$ is

(1) $(0,0)$ (2) $(0,1)$ (3) $(1,0)$ (4) $(1,1)$

SUMMARY

- If $y = f(x)$, then $\frac{dy}{dx}$ represents instantaneous rate of change of y with respect to x.
- If $y = f(g(t))$, then $\frac{dy}{dt} = f'(g(t)) \cdot g'(t)$ which is called the chain rule.
- The equation of tangent at (a,b) to the curve $y = f(x)$ is given by $y - b = \left(\frac{dy}{dx}\right)_{(a,b)} (x-a)$ or $y - b = f'(a)(x-a)$.
- Rolle's Theorem

  Let $f(x)$ be continuous in a closed interval $[a,b]$ and differentiable on the open interval (a,b). If $f(a) = f(b)$, then there is at least one point $c \in (a,b)$ where $f'(c) = 0$.
- Lagrange's Mean Value Theorem

  Let $f(x)$ be continuous in a closed interval $[a,b]$ and differentiable on the open interval (a,b) (where $f(a)$ and $f(b)$ are not necessarily equal). Then there is at least one point $c \in (a,b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.
- Taylor's series

  Let $f(x)$ be a function infinitely differentiable at $x = a$. Then $f(x)$ can be expanded as a series in an interval $(x - a, x + a)$, of the form

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{\lfloor n} (x-a)^n = f(a) + \frac{f'(a)}{\lfloor 1}(x-a) + \cdots + \frac{f^{(n)}(a)}{\lfloor n}(x-a)^n + \cdots$$

- Maclaurin's series

  In the Taylor's series if $a = 0$, then the expansion takes the form

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{\lfloor n} (x)^n = f(0) + \frac{f'(0)}{\lfloor 1}(x) + \cdots + \frac{f^{(n)}(0)}{\lfloor n}(x^n) + \cdots$$

- The l'Hôpital's rule

  Suppose $f(x)$ and $g(x)$ are differentiable functions and $g'(x) \neq 0$ with

$$\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x). \text{ Then } \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

$$\lim_{x \to a} f(x) = \pm\infty = \lim_{x \to a} g(x). \text{ Then } \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

- If the function $f(x)$ is differentiable in an open interval (a,b) then we say, if $\frac{d}{dx}(f(x)) > 0$, $\forall x \in (a,b)$ then $f(x)$ is strictly increasing in the interval (a,b).

if $\frac{d}{dx}(f(x)) < 0, \forall x \in (a,b)$ then $f(x)$ is strictly decreasing in the interval (a,b)

- A procedure for finding the absolute extrema of a continous function $f(x)$ on a closed interval $[a,b]$.

  Step 1 : Find the critical numbers of $f(x)$ in (a,b).

  Step 2 : Evaluate $f(x)$ at all critical numbers and at the endpoints a and b.

  Step 3 : The largest and the smallest of the values in Step 2 is the absolute maximum and absolute minimum of $f(x)$ respectively on the closed interval $[a,b]$.

- First Derivative Test

  Let $(c, f(c))$ be a critical point of function $f(x)$ that is continuous on an open interval I containing c. If $f(x)$ is differentiable on the interval, except possibly at c, then $f(c)$ can be classified as follows:(when moving across I from left to right)

  (i) If $f'(x)$ changes from negative to positive at c, then $f(x)$ has a local minimum $f(c)$.

  (ii) If $f'(x)$ changes from positive to negative at c, then $f(x)$ has a local maximum $f(c)$.

  (iii) If $f'(x)$ is positive on both sides of c, or negative on both sides of c then $f(x)$ has neither a local minimum nor a local minimum.

- Second Derivative Test

  Suppose that c is a critical point at which $f'(c) = 0$, that $f''(x)$ exists in a neighbourhood of c, and that $f'(c)$ exists. Then f has a relative maximum value at c if $f''(c) < 0$ and a relative minimum value at c if $f''(c) > 0$. If $f''(c) = 0$, the test is not informative.

ICT CORNER

https://ggbm.at/dy9kwgbt or Scan the QR Code

Open the Browser, type the URL Link given (or) Scan the QR code. GeoGebra work book named "**12th Standard Mathematics Vol-2**" will open. In the left side of work book there are chapters related to your text book. Click on the chapter named "**Applications of Differential Calculus**". You can see several work sheets related to the chapter. Go through all the work sheets.

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Chapter 8

Differentials and Partial Derivatives

N5B3M4

"He who hasn't tasted bitter things hasn't earned sweet things"
- Gottfried Wilhelm Leibniz

8.1 Introduction

Motivation

In real life we have to deal with many functions. Many times we have to estimate the change in the function due to change in the independent variable. Here are some real life situations.

- Suppose that a thin circular metal plate is heated uniformly. Then it's radius increases and hence its area also increases. Suppose we can measure the approximate increase in the radius. How can we estimate the increase in the area of a circular plate?
- Suppose water is getting filled in water tank that is in the shape of an inverted right circular cone. In this process the height of the water level changes, the radius of the water level changes and the volume of the water in the tank changes as time changes. In a small interval of time, if we can measure the change in the height, change in the radius, how can we estimate the change in the volume of the water in the interval?
- A satellite is launched into the space from a launch pad. A camera is being set up, to observe the launch, at a safe distance from the launch pad. As the satellite lifts up, camera's angle of elevation changes. If we know the two consecutive angles of elevation, within a small interval of time, how can we estimate the distance traveled by the satellite during that short interval of time?

To address these type of questions, we shall use the ideas of derivatives and partial derivatives to find linear approximations and differentials of the functions involved.

In the earlier chapters we have learnt the concept of derivative of a real-valued function of a single real variable. We have also learnt its applications in finding extremum of a function on its domain, and sketching the graph of a function. In this chapter, we shall see one more application of the derivative in estimating values of a function at some point. We know that linear functions, $y = mx + b$, are easy to work with; whereas nonlinear functions are computationally a bit tedious to work with.

Godfried W Leibniz
(1646 - 1716)

For instance, if we have two functions, say $f(x)=\sqrt{x+1}$, $g(x)=2x-7$ and suppose that we want to evaluate these functions at say $x=3.25$. Which one will be easy to evaluate? Obviously, $g(3.25)$ will be easier to calculate than $f(3.25)$. If we are ready to accept some error in calculating $f(3.25)$, then we can find a linear function that approximates f near $x=3$ and use this linear function to obtain an approximate value of $f(3.25)$.We know that the graph of a function is a nonvertical line if and only if it is a linear function. Out of infinitely many straight lines passing through any given point on the graph of the function, only tangent line gives a good approximation to the function, because the graph of f looks approximately a straight line on the vicinity of the point (3,2).

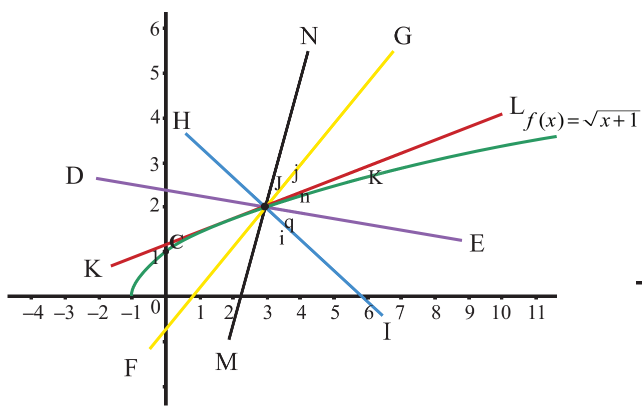

Fig. 8.1

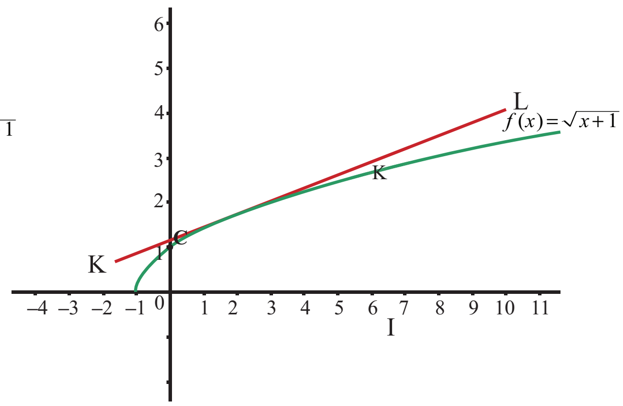

Fig. 8.2 Tangent Line

From the figures above it is clear that among these straight lines, only the tangent line to the graph of $f(x)$ at $x=3$ gives a good approximation near the point $x=3$. Basically we are **"linearizing"** the given function at a selected point $(3,2)$. This idea helps us in estimating the change in the function value near the chosen point through the change in the input. We shall use **"derivative"** to introduce the concept of **"differential"** which approximates the change in the function and will also be useful in calculating approximate values of a function near a chosen point. The derivative measures the instantaneous rate of change where as the differential approximates the change in the function values. Also, differentials are useful later in solving differential equations and evaluating definite integrals by the substitution method.

After learning differentials, we will focus on real valued functions of several variables. For functions of several variables, we shall introduce **"partial derivatives"**, a generalization of the concept of **"derivative"** of real-valued function of one variable. Why should we consider functions of more than one variable? Let us consider a simple situation that will explain the need. Suppose that a company is producing say pens and notebooks. This company is interested in maximizing its profit; then it has to find out the production level that will give maximum profit. To determine this, it has to analyze its revenue, cost, and profit functions, which are, in this case, functions of two variables (pen, notebook). Similarly, if we want to consider the volume of a box, then it will be a function of three variables namely length, width, and height. Also, the economy of a country depends on so many sectors and hence it depends on many variables. Thus it is necessary and important to consider functions involving more than one variable and develop the "concept of derivative" for functions of more than one variable. We shall also develop the concept of **"differential"** for functions of two and three variables and consider some of its applications. In this chapter, we shall consider only real-valued functions.

Learning Objectives

Upon completion of this chapter, students will be able to

- calculate the linear approximation of a function of one variable at a point
- approximate the value of a function using its linear approximation without calculators
- calculate the differential of a function
- apply linear approximation, differential in problems from real life situations
- find partial derivatives of a function of more than one variable
- calculate the linear approximation of a function of two or more variables
- determine if a given function of several variables is homogeneous or not
- apply Euler's theorem for homogeneous functions.

8.2 Linear Approximation and Differentials

8.2.1 Linear Approximation

In this section, we introduce linear approximation of a function at a point. Using the linear approximation, we shall estimate the function value near a chosen point. Then we shall introduce differential of a real-valued function of one variable, which is also useful in applications.

Let $f:(a,b)\to\mathbb{R}$ be a differentiable function and $x\in(a,b)$. Since f is differentiable at x, we have

$$\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = f'(x) \qquad \text{... (1)}$$

If Δx is small, then by (1) we have

$$f(x+\Delta x)-f(x) \approx f'(x)\Delta x\,; \qquad \text{... (2)}$$

V5Q3D6

which is equivalent to

$$f(x+\Delta x) \approx f(x)+f'(x)\Delta x, \qquad \text{... (3)}$$

where $\approx$ means **"approximately"** equal. Also, observe that as the independent variable changes from x to $x+\Delta x$, the function value changes from $f(x)$ to $f(x+\Delta x)$. Hence if Δx is small and the change in the output is denoted by Δf or Δy, then (2) can be rewritten as

$$\text{change in the output } = \Delta y = \Delta f = f(x+\Delta x)-f(x)\approx f'(x)\Delta x.$$

Note that (3) helps in approximating the value of $f(x+\Delta x)$ using $f(x)$ and $f'(x)\Delta x$. Also, for a fixed x_0, $y(x)=f(x_0)+f'(x_0)(x-x_0), x\in\mathbb{R}$, gives the tangent line for the graph of f at $(x_0, f(x_0))$ which gives a good approximation to the function f near x_0. This leads us to define

Definition 8.1 (Linear Approximation)

Let $f:(a,b)\to\mathbb{R}$ be a differentiable function and $x_0\in(a,b)$. We define the linear approximation L of f at x_0 by

$$L(x) = f(x_0)+f'(x_0)(x-x_0), \quad \forall x\in(a,b) \qquad \text{... (4)}$$

Note that by (3) and (4) we see that

$$f(x+\Delta x) \approx f(x)+f'(x)\Delta x,$$

which is useful in approximating the value of $f(x+\Delta x)$.

Note that linear approximation for f at x_0 gives a good approximation to $f(x)$ if x is close to x_0, because

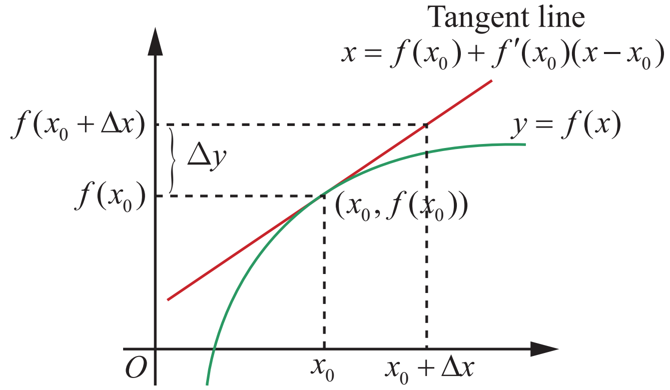

Fig. 8.3
Linear Approximation by Tangent Line

$$\text{Error} = f(x)-L(x) = f(x)-f(x_0)-f'(x_0)(x-x_0) \quad \text{... (5)}$$

approaches zero as x approaches to x_0 by continuity of f at x_0. Also, if $f(x)=mx+c$, then its linear approximation is $L(x)=(mx_0+c)+m(x-x_0)=mx+c=f(x)$, for any point $x\in(a,b)$. That is, the linear approximation, in this case, is the original function itself (is it not surprising?).

Example 8.1

Find the linear approximation for $f(x)=\sqrt{1+x}, x\geq -1$, at $x_0=3$. Use the linear approximation to estimate $f(3.2)$.

Solution

We know from (4), that $L(x)=f(x_0)+f'(x_0)(x-x_0)$. We have $x_0=3, \Delta x=0.2$ and hence $f(3)=\sqrt{1+3}=2$. Also,

$$f'(x) = \frac{1}{2\sqrt{1+x}} \text{ and hence } f'(3)=\frac{1}{2\sqrt{1+3}}=\frac{1}{4}.$$

$$\text{Thus, } L(x) = 2+\frac{1}{4}(x-3)=\frac{x}{4}+\frac{5}{4} \text{ gives the required linear approximation.}$$

$$\text{Now, } f(3.2) = \sqrt{4.2}\approx L(3.2)=\frac{3.2}{4}+\frac{5}{4}=2.050.$$

Actually, if we use a calculator to calculate we get $\sqrt{4.2}=2.04939$. ■

8.2.2 Errors: Absolute Error, Relative Error, and Percentage Error

When we are approximating a value, there occurs an error. In this section, we consider the error, which occurs by linear approximation, given by (4). We shall consider different types of errors. Taking $h=x-x_0$, we get $x=x_0+h$, then (5) becomes

$$E(h) = f(x_0+h)-f(x_0)-f'(x_0)h. \quad \text{... (6)}$$

Note that $E(0)=0$ and as we have already observed $\lim_{h\to 0}E(h)=0$ follows from the continuity of f at x_0. In addition, if f is differentiable, then from (1), it follows that

$$\lim_{h\to 0}\frac{E(h)}{h} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} - f'(x) = 0 .$$

Thus when f is differentiable at x_0, then the above equation shows that $E(h)$ actually approaches zero faster than h approaching zero. Now, we define

Definition 8.2

Suppose that certain quantity is to be determined. It's exact value is called the **actual value**. Some times we obtain its **approximate value** through some approximation process. In this case, we define

Absolute error = Actual value – Approximate value.

So (6) gives the absolute error that occurs by a linear approximation. Let us look at an example illustrating the use of linear approximation.

Example 8.2

Use linear approximation to find an approximate value of $\sqrt{9.2}$ without using a calculator.

Solution

We need to find an approximate value of $\sqrt{9.2}$ using linear approximation. Now by (3), we have $f(x_0+\Delta x) \approx f(x_0) + f'(x_0)\Delta x$. To do this, we have to identify an appropriate function f, a point x_0 and Δx. Our choice should be such that the right side of the above approximate equality, should be computable without the help of a calculator. So, we choose $f(x)=\sqrt{x}, x_0 = 9$ and $\Delta x = 0.2$. Then, $f'(x_0) = \dfrac{1}{2\sqrt{9}}$ and hence

$$\sqrt{9.2} \approx f(9) + f'(9)(0.2) = 3 + \frac{0.2}{6} = 3.03333 .$$

Now if we use a calculator, just to compare, we find $\sqrt{9.2} = 3.03315$. We see that our approximation is accurate to three decimal places and the error is $3.03315 - 3.03333 = -0.00018$. [Also note that one could choose $f(x) = \sqrt{1+x}, x_0 = 8$ and $\Delta x = 0.2$. So the choice of f and x_0 are not necessarily unique].

So in the above example, the absolute error is $3.03315 - 3.03333 = -0.00018$. Note that the absolute error says how much the error; but it does not say how good the approximation is. For instance, let us consider two simple cases.

Case 1 : Suppose that the actual value of something is 5 and its approximated value is 4, then the absolute error is $5-4=1$.

Case 2 : Suppose that the actual value of something is 100 and its approximated value is 95. In this case, the absolute error is $100-95=5$. So the absolute error in the first case is smaller when compared to the second case.

Among these two approximations, which is a better approximation; and why? The absolute error does not give a clear picture about whether an approximation is a good one or not. On the other hand, if we calculate relative error or percentage of error (defined below), it will be easy to see how good an approximation is. If the actual value is zero, then we do know how close our approximate answer is to the actual value. So if the actual value is not zero, then we define,

Definition 8.3

If the actual value is not zero, then

$$\text{Relative error} = \frac{\text{Actual value} - \text{Approximate value}}{\text{Actual value}}$$

$$\text{Percentage error} = \text{Relative error} \times 100 \quad .$$

Note: Absolute error has unit of measurement where as relative error and percentage error are units free.

Note that, in the case of the above examples,

In the first case

The relative error $= \frac{1}{5} = 0.2$; and the percentage error $= \frac{1}{5} \times 100 = 20\%$.

In the second case

The relative error $= \frac{5}{100}$; and the percentage error $= \frac{5}{100} \times 100 = 5\%$.

So the second approximation is a better approximation than the first one. Note that, in order to calculate the relative error or the percentage error one should know the actual value of what we are approximating.

Let us consider some examples.

Example 8.3

Let us assume that the shape of a soap bubble is a sphere. Use linear approximation to approximate the increase in the surface area of a soap bubble as its radius increases from 5 cm to 5.2 cm. Also, calculate the percentage error.

Solution

Recall that surface area of a sphere with radius r is given by $S(r) = 4\pi r^2$. Note that even though we can calculate the exact change using this formula, we shall try to approximate the change using the linear approximation. So, using (4), we have

$$\text{Change in the surface area} = S(5.2) - S(5) \approx S'(5)(0.2)$$

$$= 8\pi(5)(0.2) = 8\pi \text{ cm}^2$$

Exact calculation of the change in the surface gives

$$S(5.2) - S(5) = 108.16\pi - 100\pi = 8.16\pi \text{ cm}^2 \quad .$$

$$\text{Percentage error} = \text{relative error} \times 100 = \frac{8.16\pi - 8\pi}{8.16\pi} \times 100 = 1.9607\%$$

Example 8.4

A right circular cylinder has radius $r = 10$ cm. and height $h = 20$ cm. Suppose that the radius of the cylinder is increased from 10 cm to 10.1 cm and the height does not change. Estimate the change in the volume of the cylinder. Also, calculate the relative error and percentage error.

Solution

Recall that volume of a right circular cylinder is given by $V = \pi r^2 h$, where r is the radius and h is the height. So we have $V(r) = \pi r^2 h = 20\pi r^2$.

$$V(10.1) - V(10) \approx \left.\frac{dV}{dt}\right|_{r=10} (10.1 - 10) = 20\pi 2(10(0.1)).$$

Thus the estimate for the change in the volume is $40\pi\ \text{cm}^3$.

Exact calculation of the volume change gives

$$V(10.1)-V(10) = 2040.2\pi - 2000\pi = 40.2\pi\ \text{cm}^3.$$

So relative error $= \dfrac{40.2\pi - 40\pi}{40.2\pi} = \dfrac{1}{201} = 0.00497$; and hence

the percentage error = relative error $\times 100 = \dfrac{1}{201}\times 100 = 0.497\%$.

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EXERCISE 8.1

1. Let $f(x)=\sqrt[3]{x}$. Find the linear approximation at $x=27$. Use the linear approximation to approximate $\sqrt[3]{27.2}$.
2. Use the linear approximation to find approximate values of

   (i) $(123)^{\frac{2}{3}}$ (ii) $\sqrt[4]{15}$ (iii) $\sqrt[3]{26}$
3. Find a linear approximation for the following functions at the indicated points.

   (i) $f(x)=x^3-5x+12,\ x_0=2$ (ii) $g(x)=\sqrt{x^2+9},\ x_0=-4$

   (iii) $h(x)=\dfrac{x}{x+1},\ x_0=1$
4. The radius of a circular plate is measured as 12.65 cm instead of the actual length 12.5 cm. find the following in calculating the area of the circular plate:

   (i) Absolute error (ii) Relative error (iii) Percentage error
5. A sphere is made of ice having radius 10 cm. Its radius decreases from 10 cm to 9.8 cm. Find approximations for the following:

   (i) change in the volume (ii) change in the surface area
6. The time T, taken for a complete oscillation of a single pendulum with length l, is given by the equation $T=2\pi\sqrt{\dfrac{l}{g}}$, where g is a constant. Find the approximate percentage error in the calculated value of T corresponding to an error of 2 percent in the value of l.
7. Show that the percentage error in the n^{th} root of a number is approximately $\dfrac{1}{n}$ times the percentage error in the number

8.2.3 Differentials

Here again, we use the derivative concept to introduce **"Differential"**. Let us take another look at (1),

$$\frac{df}{dx} = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = f'(x) = \lim_{\Delta x\to 0}\frac{\Delta f}{\Delta x}. \qquad \text{...(7)}$$

Here $\dfrac{df}{dx}$ is a notation, used by Leibniz, for the limit of the difference quotient, which is called the **differential coefficient** of y with respect to x. Will it be meaningful to treat $\dfrac{df}{dx}$ as a quotient of df and dx? In other words, is it possible to assign meaning to df and dx so that derivative is equal to the quotient of df and dx .Well, in some cases yes. For instance, if $f(x)=mx+c, m, c$ are constants, then, $y=f(x)$.

$$\Delta y = f(x+\Delta x)-f(x)=m\Delta x=f'(x)\Delta x \quad \text{for all } x\in\mathbb{R} \quad \text{and } \Delta x$$

and hence equality in both (2), and (3). In this case changes in x and $y(=f)$ are taking place along straight lines, in which case we have,

$$\frac{\text{change in } f}{\text{change in } x} = \frac{\Delta y}{\Delta x}=f'(x)=\frac{df}{dx}=\frac{dy}{dx}.$$

Thus in this case the derivative $\dfrac{df}{dx}$ is truly a quotient of df and dx, if we take $df=\Delta f=dy$ and $dx=\Delta x$. This leads us to define the differential of f as follows:

Definition 8.4

Let $f:(a,b)\to\mathbb{R}$ be a differentiable function, for $x\in(a,b)$ and Δx the increment given to x, we define the differential of f by

$$df = f'(x)\Delta x \,. \qquad \text{... (8)}$$

First we note that if $f(x)=x$, then by (8) we get $dx=f'(x)\Delta x=1\Delta x$ which means that the differential $dx=\Delta x$, which is the change in x-axis. So the differential given by (8) is same as $df=f'(x)dx$.

Next we explore the differential for an arbitrary differentiable function $y=f(x)$. Then $\Delta f=f(x+dx)-f(x)$ gives the change in output along the graph of $y=f(x)$ and $f'(x)$ gives the slope of the tangent line at $(x, f(x))$. Let dy or df denote the increment in f along the tangent line. Then by the above observation, we have $dy=f'(x)dx$.

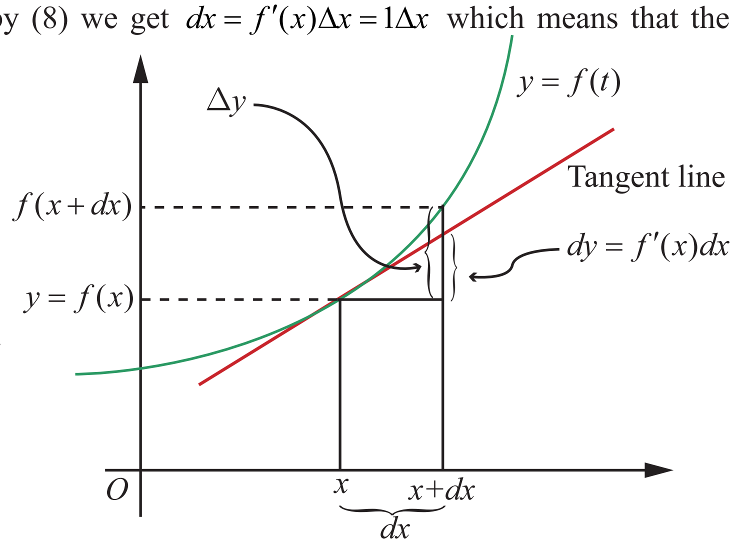

Fig. 8.4
Linear Approximation and Differential

From the figure it is clear that $\Delta f\approx dy=df=f'(x)dx$ and hence $f'(x)$ can be viewed approximately as the quotient of Δf and Δx. So we may interpret $\dfrac{df}{dx}$ as the quotient of df and dx.

Remark

We know that derivative of a function is again a function. On the other hand, differential df of a function f is not only a function of the independent variable but also depends on the change in the input namely $dx=\Delta x$. So df is a function of two changing quantities namely x and dx. Observe that $\Delta f\approx df$, which can be observed from the Fig. 8.4.

In the table below we give some functions, their derivatives and their differentials side by side for comparative purpose.

S. No.	Function	Derivative	Differential
1	$f(x)=x^n$	$f'(x)=nx^{n-1}$	$df=nx^{n-1}dx$
2	$f(x)=\cos(x^2+7x)$	$f'(x)=-\sin(x^2+7x)(2x+7)$	$df=-\sin(x^2+7x)(2x+7)dx$
3	$f(x)=\cot(x^2)$	$f'(x)=-\operatorname{cosec}^2(x^2)2x$	$df=-\operatorname{cosec}^2(x^2)2x\,dx$
4	$f(x)=\sin^{-1}(x)$	$f'(x)=\dfrac{1}{\sqrt{1-x^2}}$	$df=\dfrac{1}{\sqrt{1-x^2}}dx$
5	$f(x)=\tan^{-1}x$	$f'(x)=\dfrac{1}{1+x^2}$	$df=\dfrac{1}{1+x^2}dx$
6	$f(x)=e^{x^3-5x+7}$	$f'(x)=e^{x^3-5x+7}(3x^2-5)$	$df=e^{x^3-5x+7}(3x^2-5)dx$
7	$f(x)=\log(x^2+1)$	$f'(x)=\dfrac{2x}{x^2+1}$	$df=\dfrac{2x}{x^2+1}dx$

Next we look at the properties of differentials. These results easily follow from the definition of differential and the rules for differentiation. We give a proof for (5) below and the other proofs are left as exercises.

Properties of Differentials

Here we consider real-valued functions of real variable.

(1) If f is a constant function, then $df=0$.

(2) If $f(x)=x$ identity function, then $df=1dx$.

(3) If f is differentiable and $c\in\mathbb{R}$, then $d(cf)=cf'(x)dx$.

(4) If f,g are differentiable, then $d(f+g)=df+dg=f'(x)dx+g'(x)dx$.

(5) If f,g are differentiable, then $d(fg)=fdg+gdf=(f(x)g'(x)+f'(x)g(x))dx$.

(6) If f,g are differentiable, then $d(f/g)=\dfrac{gdf-fdg}{g^2}=\dfrac{g(x)f'(x)-f(x)g'(x)}{g^2(x)}dx$, where $g(x)\neq 0$.

(7) If f,g are differentiable and $h=f\circ g$ is defined, then $dh=f'(g(x))g'(x)dx$.

(8) If $h(x)=e^{f(x)}$, then $dh=e^{f(x)}f'(x)dx$.

(9) If $f(x)>0$ for all x and $g(x)=\log(f(x))$, then $dg=\dfrac{f'(x)}{f(x)}dx$.

Example 8.5

Let $f, g : (a,b) \to \mathbb{R}$ be differentiable functions. Show that $d(fg) = fdg + gdf$.

Solution

Let $f, g : (a,b) \to \mathbb{R}$ be differentiable functions and $h(x) = f(x)g(x)$. Then h, being a product of differentiable functions, is differentiable on (a,b). So by definition $dh = h'(x)dx$. Now by using product rule we have $h'(x) = f(x)g'(x) + f'(x)g(x)$.

Thus
$$dh = h'(x)dx = (f(x)g'(x) + f'(x)g(x))dx = f(x)g'(x)dx + f'(x)g(x)dx$$
$$= f(x)dg + g(x)df = fdg + gdf$$ ■

Example 8.6

Let $g(x) = x^2 + \sin x$. Calculate the differential dg.

Solution

Note that g is differentiable and $g'(x) = 2x + \cos x$.

Thus $dg = (2x + \cos x)dx$. ■

Example 8.7

If the radius of a sphere, with radius 10 cm, has to decrease by 0.1 cm, approximately how much will its volume decrease?

Solution

We know that volume of a sphere is given by $V = \frac{4}{3}\pi r^3$, where $r > 0$ is the radius. So the differential $dV = 4\pi r^2 dr$ and hence

$$\Delta V \approx dV = 4\pi(10)^2(9.9\text{-}10)\,\text{cm}^3$$
$$= 4\pi 10^2(-0.1)\,\text{cm}^3$$
$$= -40\pi\,\text{cm}^3.$$ ■

Note that we have used $dr = (9.9 - 10)$ cm, because radius decreases from 10 to 9.9. Again the negative sign in the answer indicates that the volume of the sphere decreases about 40π cm^3.

EXERCISE 8.2

1. Find differential dy for each of the following functions :

   (i) $y = \dfrac{(1-2x)^3}{3-4x}$ (ii) $y = (3 + \sin(2x))^{2/3}$ (iii) $y = e^{x^2-5x+7}\cos(x^2-1)$

2. Find df for $f(x) = x^2 + 3x$ and evaluate it for

   (i) $x = 2$ and $dx = 0.1$ (ii) $x = 3$ and $dx = 0.02$

3. Find Δf and df for the function f for the indicated values of $x, \Delta x$ and compare

   (i) $f(x) = x^3 - 2x^2; x = 2, \quad \Delta x = dx = 0.5$

   (ii) $f(x) = x^2 + 2x + 3;\ x = -0.5, \quad \Delta x = dx = 0.1$

4. Assuming $\log_{10} e = 0.4343$, find an approximate value of $\log_{10} 1003$.

5. The trunk of a tree has diameter 30 cm. During the following year, the circumference grew 6 cm.

   (i) Approximately, how much did the tree's diameter grow?

   (ii) What is the percentage increase in area of the tree's cross-section?

6. An egg of a particular bird is very nearly spherical. If the radius to the inside of the shell is 5 mm and radius to the outside of the shell is 5.3 mm, find the volume of the shell approximately.

7. Assume that the cross section of the artery of human is circular. A drug is given to a patient to dilate his arteries. If the radius of an artery is increased from 2 mm to 2.1 mm, how much is cross-sectional area increased approximately?

8. In a newly developed city, it is estimated that the voting population (in thousands) will increase according to $V(t) = 30 + 12t^2 - t^3,\ 0 \le t \le 8$ where t is the time in years. Find the approximate change in voters for the time change from 4 to $4\frac{1}{6}$ year.

9. The relation between the number of words y a person learns in x hours is given by $y = 52\sqrt{x},\ 0 \le x \le 9$. What is the approximate number of words learned when x changes from

   (i) 1 to 1.1 hour? (ii) 4 to 4.1 hour?

10. A circular plate expands uniformly under the influence of heat. If it's radius increases from 10.5 cm to 10.75 cm, then find an approximate change in the area and the approximate percentage change in the area.

11. A coat of paint of thickness 0.2 cm is applied to the faces of a cube whose edge is 10 cm. Use the differentials to find approximately how many cubic centimeters of paint is used to paint this cube. Also calculate the exact amount of paint used to paint this cube.

8.3 Functions of Several Variables

Recall that given a function f of x; we sketch the graph of $y = f(x)$ to understand it better. Generally, the graph of $y = f(x)$ gives a curve in the xy-plane. Also, the derivative $f'(a)$ of f at $x = a$ represents the slope of the tangent at $x = a$, to the graph of f. In the introduction we have seen the need for considering functions of more than one variable. Here we shall develop some concepts to understand functions of more than one variable. First we shall consider functions of two variables. Let $F(x, y)$ be a function of x and y. To obtain graph F, we graph $z = F(x, y)$ in the xyz-space. Also, we shall develop the concepts of continuity, partial derivatives of a function of two variables.

Let us look at an example, $g(x, y) = 30 - x^2 - y^2$, for $x, y \in \mathbb{R}$. Given a point $(x, y) \in \mathbb{R}^2$, then $z = 30 - x^2 - y^2$ gives the z coordinate of the point on the graph. Thus the point $(x, y, 30 - x^2 - y^2)$ lies $30 - x^2 - y^2$ high just above the point (x, y) in xy-plane. For instance, for $(2,3) \in \mathbb{R}^2$, the point $(2,3,30 - 2^2 - 3^2) = (2,3,17)$ lies on the graph of g. If we fix the value $y = 3$, then $g(x,3) = -x^2 + 21$ which is a function that depends only on x variable; so its graph must be a curve. Similarly, if we fix value $x = 2$, then we have $g(2, y) = 26 - y^2$ which is a function that depends only on y. In each case the graph, as the resulting function being quadratic, will be a parabola. The surface we obtain from $z = g(x, y)$ is called paraboloid.

Note that $g(x,3) = 21 - x^2$ represents a parabola; which is obtained by intersecting the surface of $z = 30 - x^2 - y^2$ with the plane $y = 3$ [see Fig. 8.5). Similarly $g(2, y) = 26 - y^2$ represents a parabola; which is obtained by intersecting the surface of $z = 30 - x^2 - y^2$ with the plane $x = 2$ [see Fig. 8.6). Following graphs describes the above discussion.

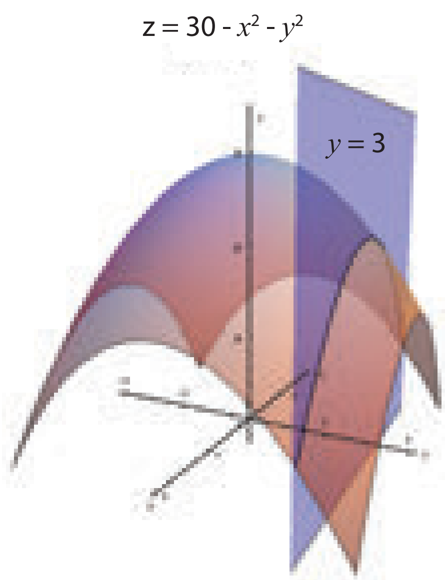

Fig. 8.5

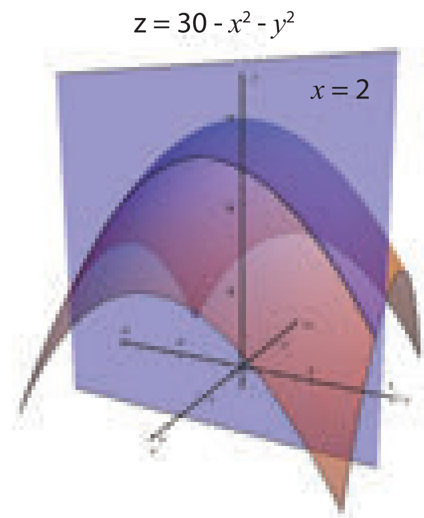

Fig. 8.6

In the same way, given a function F of a two variables say x, y, we can visualize it in the three space by considering the equation $z = F(x, y)$. Generally, this will represent a surface in $\mathbb{R}^3$.

8.3.1 Recall of Limit and Continuity of Functions of One Variable

Next we shall look at continuity of a function of two variables. Before that, it will be beneficial for us to recall the continuity of a function of single variable. We have seen the following definition of continuity in XI Std.

A function $f : (a,b) \to \mathbb{R}$ is said to be continuous at a point $x_0 \in (a,b)$ if the following hold:

(1) f is defined at x_0. (2) $\lim\limits_{x \to x_0} f(x) = L$ exists (3) $L = f(x_0)$

The key idea in the continuity lies in understanding the second condition given above. We write $\lim\limits_{x \to x_0} f(x) = L$ whenever the value $f(x)$ gets closer and closer to L as x gets closer and closer to x_0.

To make it clear and precise, let us rewrite the second condition in terms of neighbourhoods. This will help us when we talk about continuity of functions of two variables.

Definition 8.5 (Limit of a Function)

Suppose that $f : (a,b) \to \mathbb{R}$ and $x_0 \in (a,b)$. We say that f has a limit L at $x = x_0$ if for every neighbourhood $(L - \varepsilon, L + \varepsilon), \varepsilon > 0$ of L, there exists a neighbourhood $(x_0 - \delta, x_0 + \delta) \subset (a,b), \delta > 0$ of x_0 such that

$$f(x) \in (L - \varepsilon, L + \varepsilon) \text{ whenever } x \in (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}.$$

The above condition in terms of neighbourhoods can also be equivalently stated using modulus (or absolute value) notation as follows:

$\forall \varepsilon > 0, \exists\ \delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - x_0| < \delta$.

This means whenever $x \neq x_0$ and is within δ distance from x_0, then $f(x)$ is within ε distance from L. Following figures explain the interplay between ε and δ .

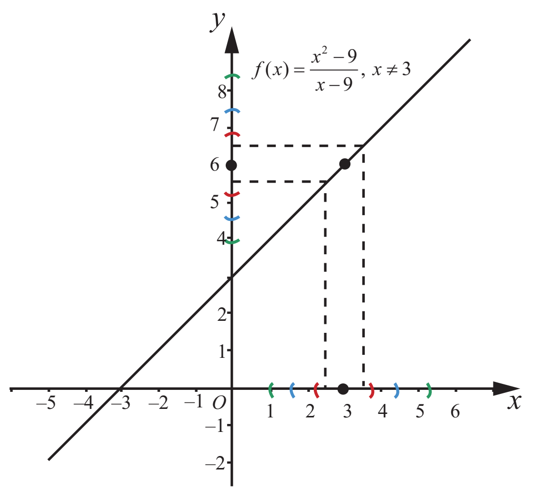

Fig. 8.7

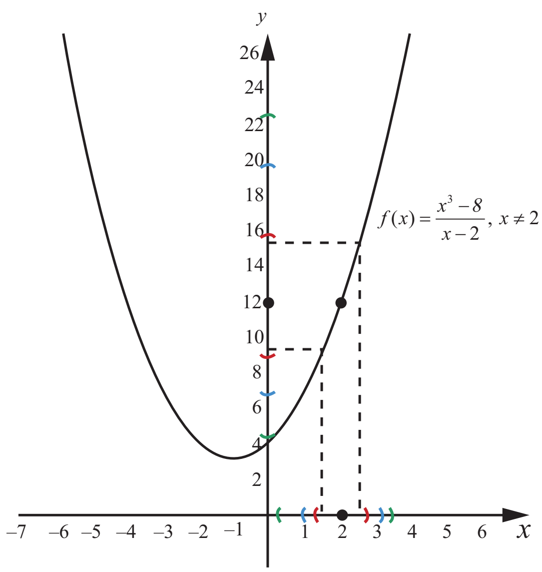

Fig. 8.8

We also know, from XI Std, that a function f defined in the neighbourhood of x_0 except possibly at x_0 has a limit at x_0 if the following hold :

(1) $\lim\limits_{x \to x_0^+} f(x) = L_1$ (right hand limit) exists (2) $\lim\limits_{x \to x_0^-} f(x) = L_2$ (left hand limit) exists

(3) $L_1 = L_2$.

Let $f(x_0) = L$ (say). Then the function f is continuous at x= x_0 if $L = L_1 = L_2$ .Note that in the limit and continuity of a single variable functions, neighbourhoods play an important role. In this case a neighbourhood of a point $x_0 \in \mathbb{R}$ looks like $(x_0 - r, x_0 + r)$, where $r > 0$. In order to develop limit and continuity of functions of two variables, we need to define neighbourhood of a point $(u, v) \in \mathbb{R}^2$. So, for $(u, v) \in \mathbb{R}^2$ and $r > 0$, a r-neighbourhood of the point (u, v) is the set

$$B_r((u, v)) = \{(x, y) \in \mathbb{R}^2 \mid (x - u)^2 + (y - v)^2 < r^2\}.$$

So a r-neighbourhood of a point (u, v) is an open disc with centre (u, v) and radius $r > 0$. If the centre is removed from the neighbourhood, then it is called a **deleted neighbourhood**.

8.4 Limit and Continuity of Functions of Two Variables

Definition 8.6 (Limit of a Function)

Suppose that $A = \{(x, y) | a < x < b, c < y < d\} \subset \mathbb{R}^2, F : A \to \mathbb{R}$. We say that F has a limit L at (u, v) if the following hold :

For every neighboourhood $(L - \varepsilon, L + \varepsilon), \varepsilon > 0$, of L, there exists a δ-neighbourhood $B_\delta((u, v)) \subset A$ of (u, v) such that $(x, y) \in B_\delta((u, v)) \setminus \{(u, v)\}, \delta > 0 \Rightarrow f(x) \in (L - \varepsilon, L + \varepsilon)$.

We denote this by $\lim\limits_{(x,y)\to(u,v)} F(x, y) = L$ if such a limit exists.

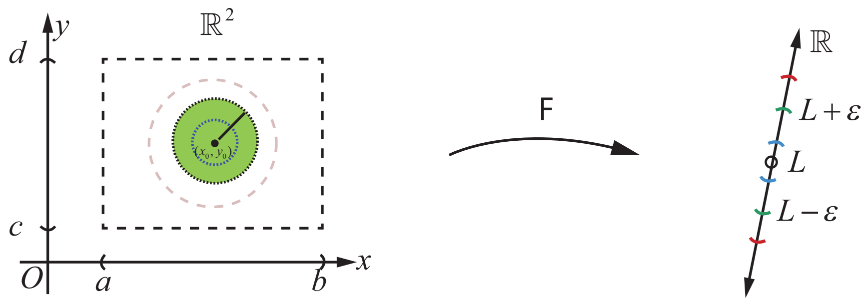

Fig. 8.9 Limit of a function

When compared to the case of a function of single variable, for a function of two variables, there is a subtle depth in the limiting process. Here the values of $F(x, y)$ should approach the same value L, as (x, y) approaches (u, v) along **every possible path to** (u, v) (including paths that are not straight lines). Fig.8.9 explains the limiting process.

All the rules for limits (limit theorems) for functions of one variable also hold true for functions of several variables.

Now, following the idea of continuity for functions of one variable, we define continuity of a function of two variables.

Definition 8.7 (Continuity)

Suppose that $A = \{(x, y) | a < x < b, c < y < d\} \subset \mathbb{R}^2, F : A \to \mathbb{R}$. We say that F is continuous at (u, v) if the following hold :

(1) F is defined at (u, v) (2) $\lim\limits_{(x,y)\to(u,v)} F(x, y) = L$ exists (3) $L = F(u, v)$.

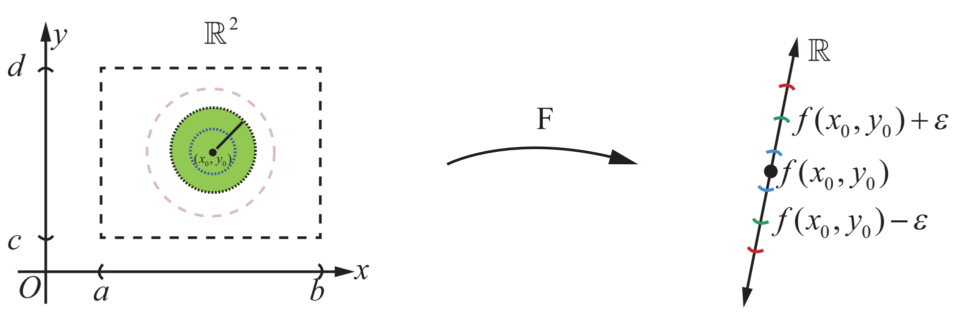

Fig. 8.10 Continuity of a function

Remark

(1) In Fig. 8.10 taking $L = F(x_0, y_0)$ will illustrate continuity at (x_0, y_0).

(2) Continuity for $f(x_1, x_2, \ldots, x_n)$ is also defined similarly as defined above.

Let us consider few examples as illustrations to understand continuity of functions of two variables.

Example 8.8

Let $f(x, y) = \dfrac{3x - 5y + 8}{x^2 + y^2 + 1}$ for all $(x, y) \in \mathbb{R}^2$. Show that f is continuous on $\mathbb{R}^2$.

Solution

Let $(a,b) \in \mathbb{R}^2$ be an arbitrary point. We shall investigate continuity of f at (a,b). That is, we shall check if all the three conditions for continuity hold for f at (a,b).

To check first condition, note that $f(a,b) = \dfrac{3a-5b+8}{a^2+b^2+1}$ is defined.

Next we want to find if $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exists or not.

So we calculate $\lim\limits_{(x,y)\to(a,b)} (3x-5y+8) = 3a-5b+8$ and $\lim\limits_{(x,y)\to(a,b)} (x^2+y^2+1) = a^2+b^2+1 \neq 0$.

Thus, by the properties of limits, we see that

$$\lim_{(x,y)\to(a,b)} f(x,y) = \frac{\lim\limits_{(x,y)\to(a,b)} (3x-5y+8)}{\lim\limits_{(x,y)\to(a,b)} (x^2+y^2+1)} = \frac{3a-5b+8}{a^2+b^2+1} = f(a,b) = \text{L exists.}$$

Now we note that $\lim\limits_{x,y\to(a,b)} f(x,y) = L = f(a,b)$. Hence f satisfies all the three conditions for continuity of f at (a,b). Since (a,b) is an arbitrary point in $\mathbb{R}^2$, we conclude that f is continuous at every point of $\mathbb{R}^2$. ■

Example 8.9

Consider $f(x,y) = \dfrac{xy}{x^2+y^2}$ if $(x,y) \neq (0,0)$ and $f(0,0) = 0$. Show that f is not continuous at $(0,0)$ and continuous at all other points of $\mathbb{R}^2$.

Solution

Note that f is defined for every $(x,y) \in \mathbb{R}^2$. First let us check the continuity at $(a,b) \neq (0,0)$. Let us say, just for instance, $(a,b) = (2,5)$. Then $f(2,5) = \dfrac{10}{29}$. Then, as in the above example, we calculate $\lim\limits_{(x,y)\to(2.5)} xy = 2(5) = 10$ and $\lim\limits_{(x,y)\to(2.5)} x^2+y^2 = 2^2+5^2 = 29 \neq 0$.

$$\text{Hence} \quad \lim_{(x,y)\to(2,5)} \frac{xy}{x^2+y^2} = \frac{10}{29}.$$

Since $f(2,5) = \dfrac{10}{29} = \lim\limits_{(x,y)\to(2,5)} \dfrac{xy}{x^2+y^2}$, it follows that f is continuous at $(2,5)$.

Exactly by similar arguments we can show that f is continuous at every point $(a,b) \neq (0,0)$. Now let us check the continuity at $(0,0)$. Note that $f(0,0) = 0$ by definition. Next we want to find if $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ exists or not.

First let us check the limit along the straight lines $y = mx$, passing through $(0,0)$.

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{mx^2}{(1+m^2)x^2} = \frac{m}{1+m^2} \neq f(0,0), \text{ if } m \neq 0.$$

So for different values of m, we get different values $\dfrac{m}{1+m^2}$ and hence we conclude that $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ **does not exist**. Hence f cannot be continuous at $(0,0)$. ■

Example 8.10

Consider $g(x,y) = \dfrac{2x^2y}{x^2+y^2}$ if $(x,y) \neq (0,0)$ and $g(0,0) = 0$. Show that g is continuous on $\mathbb{R}^2$.

Solution

Observe that the function g is defined for all $(x,y) \in \mathbb{R}^2$. It is easy to check, as in the above examples, that g is continuous at all point $(x,y) \neq (0,0)$. Next we shall check the continuity of g at $(0,0)$. For that we see if g has a limit L at $(0,0)$ and if $L = g(0,0) = 0$. So we consider

$$|g(x,y) - g(0,0)| = \left|\frac{2x^2y}{x^2+y^2} - 0\right| = \frac{2|x^2y|}{|x^2+y^2|} = \frac{2|xy||x|}{x^2+y^2} \leq \frac{(x^2+y^2)|x|}{x^2+y^2} \leq |x| \qquad \dots (9)$$

Note that in the final step above we have used $2|xy| \leq x^2 + y^2$ (which follows by considering $0 \leq (x-y)^2$) for all $x, y \in \mathbb{R}$. Note that $(x,y) \to (0,0)$ implies $|x| \to 0$. Then from (9) it follows that $\lim\limits_{(x,y)\to(0,0)} \dfrac{2x^2y}{x^2+y^2} = 0 = g(0,0)$; which proves that g is continuous at $(0,0)$. So g is continuous at every point of $\mathbb{R}^2$. ■

EXERCISE 8.3

1. Evaluate $\lim\limits_{(x,y)\to(1,2)} g(x,y)$, if the limit exists, where $g(x,y) = \dfrac{3x^2 - xy}{x^2+y^2+3}$.

2. Evaluate $\lim\limits_{(x,y)\to(0,0)} \cos\left(\dfrac{x^3+y^2}{x+y+2}\right)$. If the limit exists.

3. Let $f(x,y) = \dfrac{y^2 - xy}{\sqrt{x} - \sqrt{y}}$ for $(x,y) \neq (0,0)$. Show that $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$.

4. Evaluate $\lim\limits_{(x,y)\to(0,0)} \cos\left(\dfrac{e^x \sin y}{y}\right)$, if the limit exists.

5. Let $g(x,y) = \dfrac{x^2y}{x^4+y^2}$ for $(x,y) \neq (0,0)$ and $f(0,0) = 0$.

   (i) Show that $\lim\limits_{(x,y)\to(0,0)} g(x,y) = 0$ along every line $y = mx, m \in \mathbb{R}$.

   (ii) Show that $\lim\limits_{(x,y)\to(0,0)} g(x,y) = \dfrac{k}{1+k^2}$ along every parabola $y = kx^2, k \in \mathbb{R}\setminus\{0\}$.

6. Show that $f(x,y) = \dfrac{x^2 - y^2}{y^2+1}$ is continuous at every $(x,y) \in \mathbb{R}^2$.

7. Let $g(x,y) = \dfrac{e^y \sin x}{x}$, for $x \neq 0$ and $g(0,0) = 1$. Show that g is continuous at $(0,0)$.

8.5 Partial Derivatives

In this section, we shall see how the concept of derivative for functions of one variable may be generalized to real-valued function of several variables. First we consider functions of two variables. Let $A = \{(x, y) | a < x < b, c < y < d\} \subset \mathbb{R}^2$, and $F : A \to \mathbb{R}$ be a real-valued function. Suppose that $(x_0, y_0) \in A$; and we are interested in finding the rate of change of F at (x_0, y_0) with respect to the change **only** in the variable x. As we have seen above $F(x, y_0)$ is a function of x alone and it will represent a curve obtained by intersecting the surface $z = F(x, y)$ with $y = y_0$ plane. So we can discuss the slope of the tangent to the curve $z = F(x, y_0)$ at $x = x_0$ by finding derivative of $F(x, y_0)$ with respect to x and evaluating it at $x = x_0$. Similarly, we can find the slope of the curve $z = F(x_0, y)$ at $y = y_0$ by finding derivative of $F(x_0, y)$ with respect to y and evaluating it at $y = y_0$. These are the key ideas that motivate us to define partial derivatives below.

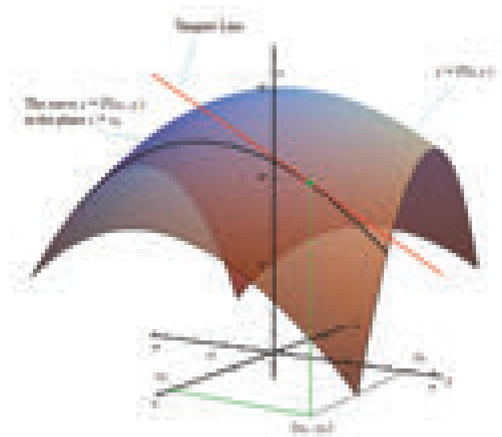

Fig. 8.11

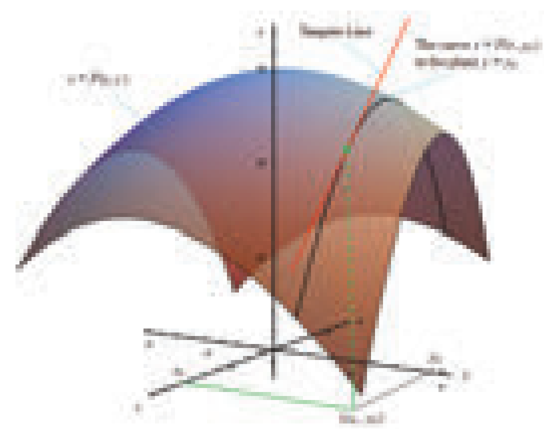

Fig. 8.12

Definition 8.8

Let $A = \{(x, y) | a < x < b, c < y < d\} \subset \mathbb{R}^2, F : A \to \mathbb{R}$ and $(x_0, y_0) \in A$.

(i) We say that F has a partial derivative with respect to x at $(x_0, y_0) \in A$ if

$$\lim_{h \to 0} \frac{F(x_0 + h, y_0) - F(x_0, y_0)}{h} \quad \text{... (10)}$$

exists. In this case, the limit value is denoted by $\frac{\partial F}{\partial x}(x_0, y_0)$.

(ii) We say F has a partial derivative with respect to y at $(x_0, y_0) \in A$ if

$$\lim_{k \to 0} \frac{F(x_0, y_0 + k) - F(x_0, y_0)}{k} \quad \text{... (11)}$$

exists. In this case, the limit value is denoted by $\frac{\partial F}{\partial y}(x_0, y_0)$.

Remarks

1. Partial derivatives for functions of three or more variables are defined exactly in a similar manner.

2. We read ∂F as **"partial F"** and ∂x as **"partial x"**. And we read $\frac{\partial F}{\partial x}$ as "partial F by partial x". It is also read as "dho F by dho x".

3. Similarly, we read $\frac{\partial F}{\partial x}$ as "partial F by partial y" or as "dho F by dho y.

4. Sometimes $\frac{\partial F}{\partial x}(x_0, y_0)$ is also denoted by $F_x(x_0, y_0)$ or $\left.\frac{\partial F}{\partial x}(x, y)\right|_{(x_0, y_0)}$

   Similarly $\frac{\partial F}{\partial y}(x_0, y_0)$ is denoted by $F_y(x_0, y_0)$, or $\left.\frac{\partial F}{\partial y}(x, y)\right|_{(x_0, y_0)}$

5. An important thing to notice is that while finding partial derivative of F with respect to x, we treat the y variable as a constant and find derivative with respect to x. That is, except for the variable with respect to which we find partial derivative, all other variables are treated as constants. That is why we call it as **"partial derivative"**.

6. If F has a partial derivative with respect to x at every point of A, then we say that $\frac{\partial F}{\partial x}(x, y)$ exists on A. Note that in this case $\frac{\partial F}{\partial x}(x, y)$ is again a real-valued function defined on A.

7. In the light of (4), it is easy to see that all the rules **(Sum, Product, Quotient, and Chain rules)** of differentiation and formulae that we have learnt earlier hold true for the partial differentiation also.

Recall that for a function of one variable, differentiability at a point always implies continuity at that point. For a function F of two variables x, y we have defined $\frac{\partial F}{\partial x}(u, v)$ and $\frac{\partial F}{\partial y}(u, v)$. Do the existence of partial derivatives of F at a point (u, v) implies continuity of F at (u, v)? Following example illustrates that this may not necessarily happen always.

Example 8.11

Let $f(x, y) = 0$ if $xy \neq 0$ and $f(x, y) = 1$ if $xy = 0$.

(i) Calculate : $\frac{\partial f}{\partial x}(0,0), \frac{\partial f}{\partial y}(0,0)$.

(ii) Show that f is not continuous at $(0,0)$.

Solution

Note that the function f takes value 1 on the x, y-axes and 0 everywhere else on $\mathbb{R}^2$. So let us calculate

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{1-1}{h} = 0;$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{k \to 0} \frac{f(0,0+k) - f(0,0)}{k} = \lim_{k \to 0} \frac{1-1}{k} = 0.$$

This completes (i).

Now for (ii) let us calculate the limit of f as $(x,y) \to (0,0)$ along the line $y = x$. Then $\lim_{(x,y)\to(0,0)} f(x,y) = 0$; because along the line $y = x$ when $x \neq 0, f(x,y) = 0$, But $f(0,0) = 1 \neq 0$; hence f cannot be continuous at $(0,0)$. ■

Example 8.12

Let $F(x,y) = x^3y + y^2x + 7$ for all $(x,y) \in \mathbb{R}^2$. Calculate $\frac{\partial F}{\partial x}(-1,3)$ and $\frac{\partial F}{\partial y}(-2,1)$.

Solution

First we shall calculate $\frac{\partial F}{\partial x}(x,y)$, then we evaluate it at $(-1,3)$. As we have already observed, we find the derivative with respect to x holding y as a constant. That is,

$$\begin{aligned}\frac{\partial F}{\partial x}(x,y) = \frac{\partial(x^3y + y^2x + 7)}{\partial x} &= \frac{\partial(x^3y)}{\partial x} + \frac{\partial(y^2x)}{\partial x} + \frac{\partial(7)}{\partial x}\\ &= 3x^2y + y^2 + 0\\ &= 3x^2y + y^2.\end{aligned}$$

So $\frac{\partial F}{\partial x}(-1,3) = 3(-1)^2 3 + 3^2 = 18$.

Next similarly we find partial derivative with respect to y.

$$\begin{aligned}\frac{\partial F}{\partial y}(x,y) = \frac{\partial(x^3y + y^2x + 7)}{\partial y} &= \frac{\partial(x^3y)}{\partial y} + \frac{\partial(y^2x)}{\partial y} + \frac{\partial(7)}{\partial y}\\ &= x^3 + 2yx + 0\\ &= x^3 + 2yx.\end{aligned}$$

Hence we have $\frac{\partial F}{\partial y}(-2,1) = (-2)^3 + 2(1)(-2) = -12$. ■

Note that in the above example $\frac{\partial F}{\partial x}(x,y) = 3x^2y + y^2$, which is again a function of two variables. So we can take the partial derivative of this function with respect to x or y. For instance, if we take $G(x,y) = 3x^2y + y^2$, then we find $\frac{\partial G}{\partial x} = 6xy$. Since $G(x,y) = \frac{\partial F}{\partial x}$, we have $\frac{\partial G}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial x}\right) = 6xy$.

We denote this as $\frac{\partial^2 F}{\partial x^2}$; which is called the second order partial derivative of F with respect to x.

Also, $\frac{\partial G}{\partial y} = 3x^2 + 2y$. Since $G(x,y) = \frac{\partial F}{\partial x}$, we have $\frac{\partial G}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = 3x^2 + 2y$. We denote this as $\frac{\partial^2 F}{\partial y \partial x}$; which is called the mixed partial derivative of F with respect to x, y. Similarly we can also calculate $\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) = 3x^2 + 2y$.

Also, if we differentiate $\frac{\partial F}{\partial y}(x,y)$ partially with respect to y we obtain $\frac{\partial^2 F}{\partial y^2}$; which is called the second order partial derivatives of F with respect to y. So for any function F defined on any subset $\{(x,y) \mid a < x < b, c < y < d\} \subset \mathbb{R}^2$ we have the following notation :

$$\frac{\partial^2 F}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial x}\right) = F_{xx}, \quad \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) = F_{xy}$$

$$\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = F_{yx}, \quad \frac{\partial^2 F}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial y}\right) = F_{yy}$$

All the above are called second order partial derivatives of F. Similarly we can define higher order partial derivatives. For example, $\frac{\partial^3 F}{\partial y^2 \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right)\right)$, and $\frac{\partial^3 F}{\partial x \partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right)\right)$.

Next we shall see more examples on partial differentiation.

Example 8.13

Let $f(x,y) = \sin(xy^2) + e^{x^3+5y}$ for all $(x,y) \in \mathbb{R}^2$. Calculate $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial y \partial x}$ and $\frac{\partial^2 f}{\partial x \partial y}$.

Solution

First we shall calculate $\frac{\partial f}{\partial x}(x,y)$. Note that f is a sum of two functions and so

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\sin(xy^2) + \frac{\partial}{\partial x}\left(e^{x^3+5y}\right) \\ &= \cos(xy^2)\frac{\partial}{\partial x}(xy^2) + e^{x^3+5y}\frac{\partial}{\partial x}(x^3+5y) \\ &= \cos(xy^2)y^2 + e^{x^3+5y}3x^2. \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\sin(xy^2) + \frac{\partial}{\partial y}\left(e^{x^3+5y}\right) \\ &= \cos(xy^2)\frac{\partial}{\partial y}(xy^2) + e^{x^3+5y}\frac{\partial}{\partial y}(x^3+5y) \\ &= \cos(xy^2)2xy + 5e^{x^3+5y}. \end{aligned}$$

Next we consider,

$$\begin{aligned} \frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(y^2\cos(xy^2) + 3x^2 e^{x^3+5y}\right) \\ &= \frac{\partial}{\partial y}(y^2\cos(xy^2)) + \frac{\partial}{\partial y}\left(3x^2 e^{x^3+5y}\right) \\ &= 2y\cos(xy^2) + y^2\left(-\sin(xy^2)2xy\right) + 3x^2 e^{x^3+5y}5 \\ &= 2y\cos(xy^2) - 2xy^3\sin(xy^2) + 15x^2 e^{x^3+5y}. \end{aligned}$$

Finally,

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(\cos(xy^2)2xy + 5e^{x^3+5y}\right)$$

$$= -\sin(xy^2)y^2 2xy + \cos(xy^2)2y + 5e^{x^3+5y}3x^2$$

$$= 2y\cos(xy^2) - 2xy^3\sin(xy^2) + 15x^2e^{x^3+5y}.$$ ■

Note that we have first used sum rule, then in the next step we have used chain rule. In the third step, product rule is used. Also, we see that $f_{xy} = f_{yx}$. Is it a coincidence? or is it always true? Actually, there are functions for which $f_{xy} \neq f_{yx}$ at some points. The following theorem gives conditions under which $f_{xy} = f_{yx}$.

Theorem 8.1 (Clairaut's Theorem)

Suppose that $A = \{(x, y) \mid a < x < b, c < y < d\} \subset \mathbb{R}^2$, $F : A \to \mathbb{R}$. If f_{xy} and f_{yx} exist in A are continuous in A, then $f_{xy} = f_{yx}$ in A.

We omit the discussion on the proof at this stage.

Example 8.14

Let $w(x, y) = xy + \frac{e^y}{y^2+1}$ for all $(x, y) \in \mathbb{R}^2$. Calculate $\frac{\partial^2 w}{\partial y \partial x}$ and $\frac{\partial^2 w}{\partial x \partial y}$.

Solution

First we calculate $\frac{\partial w}{\partial x}(x, y) = \frac{\partial(xy)}{\partial x} + \frac{\partial\left(\frac{e^y}{y^2+1}\right)}{\partial x}$.

This gives $\frac{\partial w}{\partial x}(x, y) = y + 0$ and hence $\frac{\partial^2 w}{\partial y \partial x}(x, y) = 1$. On the other hand,

$$\frac{\partial w}{\partial y}(x, y) = \frac{\partial(xy)}{\partial y} + \frac{\partial\left(\frac{e^y}{y^2+1}\right)}{\partial y}.$$

$$= x + \frac{(y^2+1)e^y - e^y 2y}{(y^2+1)^2}.$$

Hence $\frac{\partial^2 w}{\partial x \partial y}(x, y) = 1$. ■

Definition 8.9

Let $A = \{(x, y) \mid a < x < b, c < y < d\} \subset \mathbb{R}^2$. A function $u : A \to \mathbb{R}^2$ is said to be **harmonic** in A if it satisfies $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \forall (x, y) \in A$. This equation is called **Laplace's** equation.

Laplace's equation occurs in the study of many natural phenomena like heat conduction, electrostatic field, fluid flows etc.

Example 8.15

Let $u(x,y)=e^{-2y}\cos(2x)$ for all $(x,y)\in\mathbb{R}^2$. Prove that u is a harmonic function in $\mathbb{R}^2$.

Solution

We need to show that u satisfies the Laplace's equation in $\mathbb{R}^2$. Observe that $u_x(x,y)=e^{-2y}(-2)\sin(2x)$ and hence $u_{xx}(x,y)=e^{-2y}(-2)(2)\cos(2x)$.

Similarly, $u_y(x,y)=e^{-2y}(-2)\cos(2x)$ and $u_{yy}(x,y)=(-2)(-2)e^{-2y}\cos(2x)$.

Thus, $u_{xx}+u_{yy}=-4e^{-2y}\cos(2x)+4e^{-2y}\cos(2x)=0$. ■

EXERCISE 8.4

1. Find the partial derivatives of the following functions at the indicated points.

   (i) $f(x,y)=3x^2-2xy+y^2+5x+2,\quad (2,-5)$

   (ii) $g(x,y)=3x^2+y^2+5x+2,\quad (1,-2)$

   (iii) $h(x,y,z)=x\sin(xy)+z^2x,\quad \left(2,\dfrac{\pi}{4},1\right)$

   (iv) $G(x,y)=e^{x+3y}\log(x^2+y^2),(-1,1)$

2. For each of the following functions find the f_x, f_y, and show that $f_{xy}=f_{yx}$.

   (i) $f(x,y)=\dfrac{3x}{y+\sin x}$ (ii) $f(x,y)=\tan^{-1}\left(\dfrac{x}{y}\right)$ (iii) $f(x,y)=\cos(x^2-3xy)$

3. If $U(x,y,z)=\dfrac{x^2+y^2}{xy}+3z^2y$, find $\dfrac{\partial U}{\partial x},\dfrac{\partial U}{\partial y}$, and $\dfrac{\partial U}{\partial z}$.

4. If $U(x,y,z)=\log(x^3+y^3+z^3)$, find $\dfrac{\partial U}{\partial x}+\dfrac{\partial U}{\partial y}+\dfrac{\partial U}{\partial z}$.

5. For each of the following functions find the g_{xy}, g_{xx}, g_{yy} and g_{yx}.

   (i) $g(x,y)=xe^y+3x^2y$ (ii) $g(x,y)=\log(5x+3y)$

   (iii) $g(x,y)=x^2+3xy-7y+\cos(5x)$

6. Let $w(x,y,z)=\dfrac{1}{\sqrt{x^2+y^2+z^2}}$, $(x,y,z)\neq(0,0,0)$. Show that $\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}+\dfrac{\partial^2 w}{\partial z^2}=0$.

7. If $V(x,y)=e^x(x\cos y-y\sin y)$, then prove that $\dfrac{\partial^2 V}{\partial x^2}+\dfrac{\partial^2 V}{\partial y^2}=0$.

8. If $w(x,y)=xy+\sin(xy)$, then prove that $\dfrac{\partial^2 w}{\partial y\partial x}=\dfrac{\partial^2 w}{\partial x\partial y}$.

9. If $v(x,y,z) = x^3 + y^3 + z^3 + 3xyz$, show that $\dfrac{\partial^2 v}{\partial y \partial z} = \dfrac{\partial^2 v}{\partial z \partial y}$.

10. A firm produces two types of calculators each week, x number of type A and y number of type B. The weekly revenue and cost functions (in rupees) are $R(x,y) = 80x + 90y + 0.04xy - 0.05x^2 - 0.05y^2$ and $C(x,y) = 8x + 6y + 2000$ respectively.

    (i) Find the profit function $P(x,y)$,

    (ii) Find $\dfrac{\partial P}{\partial x}(1200,1800)$ and $\dfrac{\partial p}{\partial y}(1200,1800)$ and interpret these results.

8.6 Linear Approximation and Differential of a function of several variables

Earlier in this chapter, we have seen that linear approximation and differential of a function of one variable. Here we introduce similar ideas for functions of two variables and three variables. In general for functions of several variables these concepts can be defined similarly.

Definition 8.10

Let $A = \{(x,y) \mid a < x < b, c < y < d\} \subset \mathbb{R}^2, F : A \to \mathbb{R}$, and $(x_0, y_0) \in A$.

(i) The linear approximation of F at $(x_0, y_0) \in A$ is defined to be

$$F(x,y) = F(x_0,y_0) + \left.\frac{\partial F}{\partial x}\right|_{(x_0,y_0)} (x - x_0) + \left.\frac{\partial F}{\partial y}\right|_{(x_0,y_0)} (y - y_0) \qquad \text{... (12)}$$

(ii) The differential of F is defined to be

$$dF = \frac{\partial F}{\partial x}(x,y)dx + \frac{\partial F}{\partial y}(x,y)dy, \qquad \text{... (13)}$$

where $dx = \Delta x$ and $dy = \Delta y$,

Here we shall outline the linear approximations and differential for the functions of three variables. Actually, we can define linear approximations and differential for real valued function having more variables, but we restrict ourselves to only three variables.

Definition 8.11

Let $A = \{(x,y,z) \mid a < x < b, c < y < d,\ e < z < f\} \subset \mathbb{R}^3, F : A \to \mathbb{R}$ and $(x_0, y_0, z_0) \in A$.

(i) The linear approximation of F at $(x_0, y_0, z_0) \in A$ is defined to be

$$F(x,y,z) = F(x_0,y_0,z_0) + \left.\frac{\partial F}{\partial x}\right|_{(x_0,y_0,z_0)} (x - x_0) + \left.\frac{\partial F}{\partial y}\right|_{(x_0,y_0,z_0)} (y - y_0) + \left.\frac{\partial F}{\partial z}\right|_{(x_0,y_0,z_0)} (z - z_0);$$

...(14)

(ii) The differential of F is defined by

$$dF = \frac{\partial F}{\partial x}(x,y,z)dx + \frac{\partial F}{\partial y}(x,y,z)dy + \frac{\partial F}{\partial z}(x,y,z)dz, \qquad \text{...(15))}$$

where $dx = \Delta x, dy = \Delta y$ and $dz = \Delta z$, Geometrically, in the case of function f of one variable, the linear approximation at a point x_0 represents the tangent line to the graph of $y = f(x)$ at x_0. Similarly, in the case of a function F of two variables, the linear approximation at a point (x_0, y_0) represents the tangent plane to the graph of $z = F(x, y)$ at (x_0, y_0).

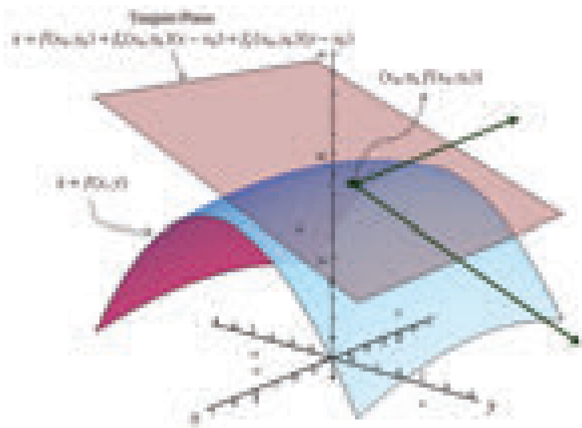

Fig. 8.13
Linear Approximation by Tangent Plane

Example 8.16

If $w(x, y, z) = x^2 y + y^2 z + z^2 x,\ x, y, z \in \mathbb{R}$, find the differential dw.

Solution

First let us find w_x, w_y, and w_z.

Now $w_x = 2xy + z^2, w_y = 2yz + x^2$ and $w_z = 2zx + y^2$.

Thus,by (15), the differential is

$$dw = (2xy + z^2)dx + (2yz + x^2)dy + (2zx + y^2)dz.$$

■

Example 8.17

Let $U(x, y, z) = x^2 - xy + 3\sin z,\ x, y, z \in \mathbb{R}$. Find the linear approximation for U at $(2, -1, 0)$.

Solution

By (14), linear approximation is given by

$$L(x, y, z) = U(x_0, y_0, z_0) + \left.\frac{\partial U}{\partial x}\right|_{(x_0, y_0, z_0)}(x - x_0) + \left.\frac{\partial U}{\partial y}\right|_{(x_0, y_0, z_0)}(y - y_0) + \left.\frac{\partial U}{\partial z}\right|_{(x_0, y_0, z_0)}(z - z_0).$$

Now $U_x = 2x - y, U_y = -x$ and $U_z = 3\cos z$.

Here $(x_0, y_0, z_0) = (2, -1, 0)$, hence $U_x(2, -1, 0) = 5,\ U_y(2, -1, 0) = -2$ and $U_z(2, -1, 0) = 3$.

Thus $L(x, y, z) = 6 + 5(x - 2) - 2(y + 1) + 3(z - 0) = 5x - 2y + 3z - 6$ is the required linear approximation for U at $(2, -1, 0)$. ■

EXERCISE 8.5

1. If $w(x, y) = x^3 - 3xy + 2y^2,\ x, y \in \mathbb{R}$, find the linear approximation for w at $(1, -1)$.

2. Let $z(x, y) = x^2 y + 3xy^4,\ x, y \in \mathbb{R}$. Find the linear approximation for z at $(2, -1)$.

3. If $v(x, y) = x^2 - xy + \frac{1}{4}y^2 + 7,\ x, y \in R$, find the differential dv.

4. Let $V(x,y,z) = xy + yz + zx,\ x,y,z \in \mathbb{R}$. Find the differential dV .

8.6.1 Function of Function Rule

Let F be a function of two variables x, y. Sometimes these variables may be functions of a single variable having same domain. In this case, the function F ultimately depends only on one variable. So we should be able to treat this F as a function of single variable and study about $\frac{dF}{dt}$. In fact, this is not a coincidence, it can be proved that

Theorem 8.2

Suppose that $W(x,y)$ is a function of two variables x, y having partial derivatives $\frac{\partial W}{\partial x}, \frac{\partial W}{\partial y}$. If both the variables x, y are differentiable functions of a single variable t, then W is a differentiable function of t and

$$\frac{dW}{dt} = \frac{\partial W}{\partial x}\frac{dx}{dt} + \frac{\partial W}{\partial y}\frac{dy}{dt} \qquad \text{...(16)}$$

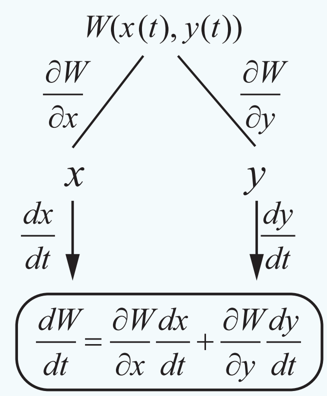

Fig. 8.14

Let us consider an example illustrating the above theorem.

C5U1E2

Example 8.18

Verify the above theorem for $F(x,y) = x^2 - 2y^2 + 2xy$ and $x(t) = \cos t,\ y(t) = \sin t, t \in [0, 2\pi]$.

Solution

Let $F(x,y) = x^2 - 2y^2 + 2xy$ and $x(t) = \cos t$, $y(t) = \sin t$.

Then $F(x,y) = \cos^2 t - 2\sin^2 t + 2\cos t \sin t$ and thus F has becomes a function of one variable t. So by using chain rule, we see that

$$\begin{aligned}\frac{dF}{dt} &= 2\cos t(-\sin t) - 4\sin t\cos t + 2(-\sin^2 t + \cos^2 t)\\ &= -6\cos t\sin t + 2(-\sin^2 t + \cos^2 t).\end{aligned}$$

On the other hand if we calculate

$$\begin{aligned}\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} &= (2x+2y)\frac{dx}{dt} + (2x-4y)\frac{dy}{dt}\\ &= 2(\cos t + \sin t)(-\sin t) + 2(\cos t - 2\sin t)(\cos t)\\ &= -6\cos t\sin t + 2(-\sin^2 t + \cos^2 t)\\ &= \frac{dF}{dt}.\end{aligned}$$

Example 8.19

Let $g(x,y)=x^2-yx+\sin(x+y),\ x(t)=e^{3t},\ y(t)=t^2,\ t\in\mathbb{R}$. Find $\dfrac{dg}{dt}$.

Solution

We shall follow the tree diagram to calculate.

So first we need to find $\dfrac{\partial g}{\partial x},\ \dfrac{\partial g}{\partial y},\ \dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$.

Now $\dfrac{\partial g}{\partial x}=2x-y+\cos(x+y),\ \dfrac{\partial g}{\partial y}=-x+\cos(x+y),\ \dfrac{dx}{dt}=3e^{3t}$ and $\dfrac{dy}{dt}=2t$.

Thus

$$\begin{aligned}\frac{dg}{dt} &= \frac{\partial g}{\partial x}\frac{dx}{dt}+\frac{\partial g}{\partial y}\frac{dy}{dt}\\ &= \left(2x-y+\cos(x+y)\right)3e^{3t}+\left(-x+\cos(x+y)\right)(2t)\\ &= (2e^{3t}-t^2+\cos(e^{3t}+t^2))3e^{3t}+(-e^{3t}+\cos(e^{3t}+t^2))(2t)\\ &= 6e^{6t}-3t^2e^{3t}+3e^{3t}\cos(e^{3t}+t^2)-2te^{3t}+2t\cos(e^{3t}+t^2).\end{aligned}$$

Also, some times our $W(x,y)$ will be such that $x=x(s,t)$, and $y=y(s,t)$ where $s,t\in\mathbb{R}$. Then W can be considered as a function that depends on s and t. If x,y both have partial derivatives with respect to s,t and W has partial derivatives with respect to x and y, then we can calculate the partial derivatives of W with respect to s and t using the following theorem.

Theorem 8.3

Suppose that $W(x,y)$ is a function of two variables x,y having partial derivatives $\dfrac{\partial W}{\partial x},\dfrac{\partial W}{\partial y}$. If both variables $x=x(s,t)$ and $y=y(s,t)$, where $s,t\in\mathbb{R}$, have partial derivatives with respect to both s and t, then

$$\frac{\partial W}{\partial s} = \frac{\partial W}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial W}{\partial y}\frac{\partial y}{\partial s}, \qquad \text{... (17)}$$

$$\frac{\partial W}{\partial t} = \frac{\partial W}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial W}{\partial y}\frac{\partial y}{\partial t}. \qquad \text{... (18)}$$

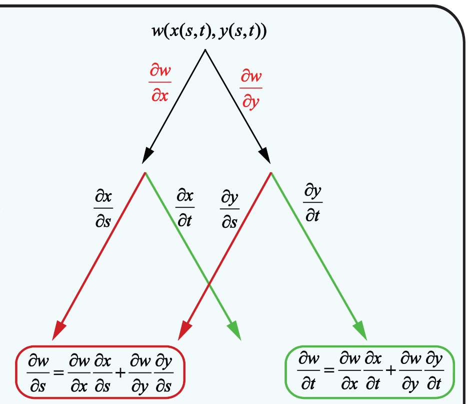

Fig. 8.15

We omit the proof. The above theorem is very useful. For instance, consider the situation in which $x=r\cos\theta$, and $y=\sin\theta, r\geq 0$ and $\theta\in\mathbb{R}$, (change from cartesian co-ordinate to polar co-ordinate system). The above theorem can be generalized for functions having n number of variables.

Let us consider an example.

Example 8.20

Let $g(x,y)=2y+x^2, x=2r-s, y=r^2+2s, r,s\in\mathbb{R}$. Find $\dfrac{\partial g}{\partial r},\dfrac{\partial g}{\partial s}$.

Solution

Here again we shall use the tree diagram to calculate $\dfrac{\partial g}{\partial r},\dfrac{\partial g}{\partial s}$

Hence we find $\quad \dfrac{\partial g}{\partial x} = 2x,\ \dfrac{\partial g}{\partial y}=2,\ \dfrac{\partial x}{\partial r}=2,\ \dfrac{\partial x}{\partial s}=-1,\ \dfrac{\partial y}{\partial r}=2r$, and $\dfrac{\partial y}{\partial s}=2$.

Now $$\frac{\partial g}{\partial r} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial r} = 2x(2) + 2(2r) = 12r - 4s .$$

also, $$\frac{\partial g}{\partial s} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial s} = 2x(-1) + (2)2 = 2s - 4r + 4 .$$ ■

EXERCISE 8.6

1. If $u(x, y) = x^2 y + 3xy^4$, $x = e^t$ and $y = \sin t$, find $\dfrac{du}{dt}$ and evaluate it at $t = 0$.

2. If $u(x, y, z) = xy^2 z^3$, $x = \sin t$, $y = \cos t$, $z = 1 + e^{2t}$, find $\dfrac{du}{dt}$.

3. If $w(x, y, z) = x^2 + y^2 + z^2$, $x = e^t$, $y = e^t \sin t$ and $z = e^t \cos t$, find $\dfrac{dw}{dt}$.

4. Let $U(x, y, z) = xyz$, $x = e^{-t}$, $y = e^{-t}\cos t$, $z = \sin t$, $t \in \mathbb{R}$. Find $\dfrac{dU}{dt}$.

5. If $w(x, y) = 6x^3 - 3xy + 2y^2$, $x = e^s$, $y = \cos s$, $s \in \mathbb{R}$, find $\dfrac{dw}{ds}$, and evaluate at $s = 0$,

6. If $z(x, y) = x\tan^{-1}(xy)$, $x = t^2$, $y = se^t$, $s, t \in \mathbb{R}$. Find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$ at $s = t = 1$.

7. Let $U(x, y) = e^x \sin y$, where $x = st^2$, $y = s^2 t$, $s, t \in \mathbb{R}$. Find $\dfrac{\partial U}{\partial s}, \dfrac{\partial U}{\partial t}$ and evaluate them at $s = t = 1$.

8. Let $z(x, y) = x^3 - 3x^2 y^3$, where $x = se^t$, $y = se^{-t}$, $s, t \in \mathbb{R}$. Find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$.

9. $W(x, y, z) = xy + yz + zx$, $x = u - v$, $y = uv$, $z = u + v$, $u, v \in \mathbb{R}$. Find $\dfrac{\partial W}{\partial u}, \dfrac{\partial W}{\partial v}$, and evaluate them at $\left(\dfrac{1}{2}, 1\right)$.

8.6.2 Homogeneous Functions and Euler's Theorem

Definition 8.12

(a) Let $A = \{(x, y) \mid a < x < b,\ c < y < d\} \subset \mathbb{R}^2$, $F : A \to \mathbb{R}$, we say that F is a homogeneous function on A, if there exists a constant p such that $F(\lambda x, \lambda y) = \lambda^p F(x, y)$ for all $\lambda \in \mathbb{R}$ and sutitably restricted $\lambda, x, y,$ such that $(\lambda x, \lambda y) \in A$. This constant p is called degree of F.

(b) Let $B = \{(x, y, z) \mid a < x < b,\ c < y < d,\ u < z < v\} \subset \mathbb{R}^3$, $G : B \to \mathbb{R}$, we say that G is a homogeneous function on B, if there exists a constant p such that $G(\lambda x, \lambda y, \lambda z) = \lambda^p G(x, y, z)$ for all $\lambda \in \mathbb{R}$ and sutitably restricted $\lambda, x, y, z,$ such that $(\lambda x, \lambda y, \lambda z) \in B$. This constant p is called degree of G.

Note: Division by any variable may occur, to avoid division by zero, we say that λ, x, y, z are sutitably restricted real numbers.

These types of functions are important in Ordinary differential equations (Chapter 10). Let us consider some examples.

Consider $F(x,y) = x^3 - 2y^3 + 5xy^2,\ (x,y) \in \mathbb{R}^2$. Then

$$F(\lambda x, \lambda y) = (\lambda x)^3 - 2(\lambda y)^3 + 5(\lambda x)(\lambda y)^2 = \lambda^3(x^3 - 2y^3 + 5xy^2)$$

and hence F is a homogeneous function of degree 3.

On the other hand,

$G(x,y) = e^{x^2} + 3y^2$ is not a homogeneous function because,

$$G(\lambda x, \lambda y) = e^{(\lambda x)^2} + 3(\lambda y)^2 \neq \lambda^p G(x,y)$$

for any $\lambda \neq 1$ and any p.

Example 8.21

Show that $F(x,y) = \dfrac{x^2 + 5xy - 10y^2}{3x + 7y}$ is a homogeneous function of degree 1.

Solution

We compute

$$F(\lambda x, \lambda y) = \frac{(\lambda x)^2 + 5(\lambda x)(\lambda y) - 10(\lambda y)^2}{3\lambda x + 7\lambda y} = \frac{\lambda^2}{\lambda}\left(\frac{x^2 + 5xy - 10y^2}{3x + 7y}\right) = \lambda F(x,y)$$

for all $\lambda \in \mathbb{R}$. So F is a homogeneous function of degree 1.

We state the following theorem of Leonard Euler on homogeneous functions. ■

Definition 8.13 (Euler)

Suppose that $A = \{(x,y) \mid a < b,\ c < y < d\} \subset \mathbb{R}^2,\ F : A \to \mathbb{R}^2$. If F is having continuous partial derivatives and homogeneous on A, with degree p, then

$$x\frac{\partial F}{\partial x}(x,y) + y\frac{\partial F}{\partial y}(x,y) = pF(x,y)\ \forall\ (x,y) \in A.$$

Suppose that $B = \{(x,y,z) \mid a < x < b,\ c < y < d,\ u < z < v\} \subset \mathbb{R}^3,\ F : B \to \mathbb{R}^3$. If F is having continuous partial derivatives and homogeneous on B, with degree p, then

$$x\frac{\partial F}{\partial x}(x,y,z) + y\frac{\partial F}{\partial y}(x,y,z) + z\frac{\partial F}{\partial z}(x,y,z) = pF(x,y,z)\ \forall\ (x,y,z) \in B.$$

We omit the proof. The above theorem is also true for any homogeneous function of n variables; and is useful in certain calculations involving first order partial derivatives.

Example 8.22

If $u = \sin^{-1}\left(\dfrac{x+y}{\sqrt{x} + \sqrt{y}}\right)$, Show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{1}{2}\tan u$.

Solution

Note that the function u is not homogeneous. So we cannot apply Euler's Theorem for u.

However, note that $f(x,y) = \dfrac{x+y}{\sqrt{x} + \sqrt{y}} = \sin u$ is homogeneous; because

$$f(tx,ty)=\frac{tx+ty}{\sqrt{tx}+\sqrt{ty}}=t^{1/2}f(x,y),\ \forall\ x,y,t\geq 0.$$

Thus f is homogeneous with degree $\frac{1}{2}$, and so by Euler's Theorem we have

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y} = \frac{1}{2}f(x,y).$$

Now substituting $f=\sin u$ in the above equation, we obtain

$$x\frac{\partial(\sin u)}{\partial x}+y\frac{\partial(\sin u)}{\partial y} = \frac{1}{2}\sin u$$

$$x\cos u\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial y} = \frac{1}{2}\sin u \qquad \ldots (19)$$

Dividing both sides by $\cos u$ we obtain

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u.$$

Note:

Solving this problem by direct calculation will be possible; but will involve lengthy calculations.

EXERCISE 8.7

1. In each of the following cases, determine whether the following function is homogeneous or not. If it is so, find the degree.

   (i) $f(x,y)=x^2y+6x^3+7$ (ii) $h(x,y)=\dfrac{6x^2y^3-\pi y^5+9x^4y}{2020x^2+2019y^2}$

   (iii) $g(x,y,z)=\dfrac{\sqrt{3x^2+5y^2+z^2}}{4x+7y}$ (iv) $U(x,y,z)=xy+\sin\left(\dfrac{y^2-2z^2}{xy}\right)$.

2. Prove that $f(x,y)=x^3-2x^2y+3xy^2+y^3$ is homogeneous; what is the degree? Verify Euler's Theorem for f.

3. Prove that $g(x,y)=x\log\left(\dfrac{y}{x}\right)$ is homogeneous; what is the degree? Verify Euler's Theorem for g.

4. If $u(x,y)=\dfrac{x^2+y^2}{\sqrt{x+y}}$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=\dfrac{3}{2}u$.

5. If $v(x,y)=\log\left(\dfrac{x^2+y^2}{x+y}\right)$, prove that $x\dfrac{\partial v}{\partial x}+y\dfrac{\partial v}{\partial y}=1$.

6. If $w(x,y,z)=\log\left(\dfrac{5x^3y^4+7y^2xz^4-75y^3z^4}{x^2+y^2}\right)$, find $x\dfrac{\partial w}{\partial x}+y\dfrac{\partial w}{\partial y}+z\dfrac{\partial w}{\partial z}$.

EXERCISE 8.8

Choose the correct or the most suitable answer from the given four alternatives :

1. A circular template has a radius of 10 cm. The measurement of radius has an approximate error of 0.02 cm. Then the percentage error in calculating area of this template is

   (1) 0.2% (2) 0.4% (3) 0.04% (4) 0.08%

2. The percentage error of fifth root of 31 is approximately how many times the percentage error in 31?

   (1) $\frac{1}{31}$ (2) $\frac{1}{5}$ (3) 5 (4) 31

3. If $u(x,y) = e^{x^2+y^2}$, then $\frac{\partial u}{\partial x}$ is equal to

   (1) $e^{x^2+y^2}$ (2) $2xu$ (3) x^2u (4) y^2u

4. If $v(x,y) = \log(e^x + e^y)$, then $\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}$ is equal to

   (1) $e^x + e^y$ (2) $\frac{1}{e^x + e^y}$ (3) 2 (4) 1

5. If $w(x,y) = x^y, x > 0$, then $\frac{\partial w}{\partial x}$ is equal to

   (1) $x^y \log x$ (2) $y \log x$ (3) yx^{y-1} (4) $x \log y$

6. If $f(x,y) = e^{xy}$, then $\frac{\partial^2 f}{\partial x \partial y}$ is equal to

   (1) xye^{xy} (2) $(1+xy)e^{xy}$ (3) $(1+y)e^{xy}$ (4) $(1+x)e^{xy}$

7. If we measure the side of a cube to be 4 cm with an error of 0.1 cm, then the error in our calculation of the volume is

   (1) 0.4 cu.cm (2) 0.45 cu.cm (3) 2 cu.cm (4) 4.8 cu.cm

8. The change in the surface area $S = 6x^2$ of a cube when the edge length varies from x_0 to $x_0 + dx$ is

   (1) $12x_0 + dx$ (2) $12x_0 dx$ (3) $6x_0 dx$ (4) $6x_0 + dx$

9. The approximate change in the volume V of a cube of side x metres caused by increasing the side by 1% is

   (1) $0.3xdx\, m^3$ (2) $0.03x\, m^3$ (3) $0.03x^2 m^3$ (4) $0.03x^3 m^3$

10. If $g(x,y) = 3x^2 - 5y + 2y^2, x(t) = e^t$ and $y(t) = \cos t$, then $\frac{dg}{dt}$ is equal to

    (1) $6e^{2t} + 5\sin t - 4\cos t \sin t$ (2) $6e^{2t} - 5\sin t + 4\cos t \sin t$

    (3) $3e^{2t} + 5\sin t + 4\cos t \sin t$ (4) $3e^{2t} - 5\sin t + 4\cos t \sin t$

11. If $f(x)=\frac{x}{x+1}$, then its differential is given by

(1) $\frac{-1}{(x+1)^2}dx$ (2) $\frac{1}{(x+1)^2}dx$ (3) $\frac{1}{x+1}dx$ (4) $\frac{-1}{x+1}dx$

12. If $u(x,y)=x^2+3xy+y-2019$, then $\left.\frac{\partial u}{\partial x}\right|_{(4,-5)}$ is equal to

(1) -4 (2) -3 (3) -7 (4) 13

13. Linear approximation for $g(x)=\cos x$ at $x=\frac{\pi}{2}$ is

(1) $x+\frac{\pi}{2}$ (2) $-x+\frac{\pi}{2}$ (3) $x-\frac{\pi}{2}$ (4) $-x-\frac{\pi}{2}$

14. If $w(x,y,z)=x^2(y-z)+y^2(z-x)+z^2(x-y)$, then $\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}+\frac{\partial w}{\partial z}$ is

(1) $xy+yz+zx$ (2) $x(y+z)$ (3) $y(z+x)$ (4) 0

15. If $f(x,y,z)=xy+yz+zx$, then f_x-f_z is equal to

(1) $z-x$ (2) $y-z$ (3) $x-z$ (4) $y-x$

SUMMARY

- Let $f:(a,b)\to\mathbb{R}$ be a differentiable function and $x_0\in(a,b)$ then linear approximation L of f at x_0 is given by
  $L(x)=f(x_0)+f'(x_0)(x-x_0)\,\forall x\in(a,b)$
- Absolute error $=$Actual value $-$ Approximate value

  Relative error $=\frac{\text{Absolute error}}{\text{Actual value}}$

  Percentage error $=$ Relative error $\times 100$

  (or)

  $\frac{\text{Absolute error}}{\text{Acutal value}}\times 100$
- Let $f:(a,b)\to\mathbb{R}$ be a differentiable function. For $x\in(a,b)$ and Δx the increment given to x, the differential of f is defined by $df=f'(x)\Delta x$.
- All the rules for limits (limit theorems) for functions of one variable also hold true for functions of several variables.
- Let $A=\{(x,y)\,|\,a<x<b,\ c<y<d\}\subset\mathbb{R}^2, F:A\to\mathbb{R}$ and $(x_0,y_0)\in A$.

(i) F has a partial derivative with respect to x at $(x_0, y_0) \in A$ if $\lim\limits_{h \to 0} \dfrac{F(x_0 + h, y_0) - F(x_0, y_0)}{h}$ exists and it is denoted by $\left.\dfrac{\partial F}{\partial x}\right|_{(x_0, y_0)}$.

F has a partial derivative with respect to y at $(x_0, y_0) \in A$ if $\lim\limits_{k \to 0} \dfrac{F(x_0, y_0 + k) - F(x_0, y_0)}{k}$ exists and limit value is defined by $\left.\dfrac{\partial F}{\partial y}\right|_{(x_0, y_0)}$.

- Clariant's Theorem: Suppose that $A = \{(x, y) \mid a < x < b,\ c < y < d\} \subset \mathbb{R}^2$, $F : A \to \mathbb{R}$. If f_{xy} and f_{yx} exist in A and are continuous in A, then $f_{xy} = f_{yx}$ in A.
- Let $A = \{(x, y) \mid a < x < b,\ c < y < d\} \subset \mathbb{R}^2$. A function $U : A \to \mathbb{R}$ is said to be harmonic in A if it satisfies $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0,\ \forall (x, y) \in A$. This equation is called Laplace's equation.
- Let $A = \{(x, y) \mid a < x < b,\ c < y < d\} \subset \mathbb{R}^2$, $F : A \to \mathbb{R}$ and $(x_0, y_0) \in A$.

  (i) The linear approximation of F at $(x_0, y_0) \in A$ is defined to be

  $$L(x, y) = F(x_0, y_0) + \left.\frac{\partial F}{\partial x}\right|_{(x_0, y_0)} (x - x_0) + \left.\frac{\partial F}{\partial y}\right|_{(x_0, y_0)} (y - y_0)$$

  (ii) The differential of F is defined to be $dF = \dfrac{\partial F}{\partial x} dx + \dfrac{\partial F}{\partial y} dy$ where $\Delta x = dx$ and $\Delta y = dy$.
- Suppose w is a function of two variables x, y where x and y are functions of a single variable 't' then $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x} \cdot \dfrac{dx}{dt} + \dfrac{\partial w}{\partial y} \cdot \dfrac{dy}{dt}$
- Suppose w is a function of two variables x and y where x and y are functions of two variables s and t then, $\dfrac{\partial w}{\partial s} = \dfrac{\partial w}{\partial x} \cdot \dfrac{\partial x}{\partial s} + \dfrac{\partial w}{\partial y} \cdot \dfrac{\partial y}{\partial s}$, $\dfrac{\partial w}{\partial t} = \dfrac{\partial w}{\partial x} \cdot \dfrac{\partial x}{\partial t} + \dfrac{\partial w}{\partial y} \cdot \dfrac{\partial y}{\partial t}$
- Suppose that $A = \{(x, y) \mid a < x < b,\ c < y < d\} \subset \mathbb{R}^2$, $F : A \to \mathbb{R}^2$. If F is having continuous partial derivatives and homogeneous on A, with degree p, then $x \dfrac{\partial F}{\partial x} + y \dfrac{\partial F}{\partial y} = pF$.

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Open the Browser, type the URL Link given (or) Scan the QR code. GeoGebra work book named "**12th Standard Mathematics Vol-2**" will open. In the left side of work book there are chapters related to your text book. Click on the chapter named "**Differentials and Partial Derivatives**". You can see several work sheets related to the chapter. Go through all the work sheets

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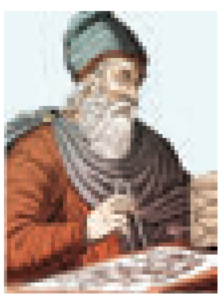
Chapter
9
Applications of Integration

"Give me a place to stand and I will move the earth"

- Archimedes

9.1.Introduction



Archimedes of Syracuse (288BC(BCE)-212BC(BCE)) was a Greek mathematician, physicist, engineer, inventor

One of the earliest mathematicians who made wonderful discoveries to compute the areas and volumes of geometrical objects was Archimedes. Archimedes proved that the area enclosed by a parabola and a straight line is $\frac{4}{3}$ times the area of an inscribed triangle (see Fig. 9.1).

He obtained the area by segmenting it into infinitely many elementary areas and then finding their sum. This limiting concept is inbuilt in the definition of definite integral which we are going to develop here and apply the same in finding areas and volumes of certain geometrical shapes.

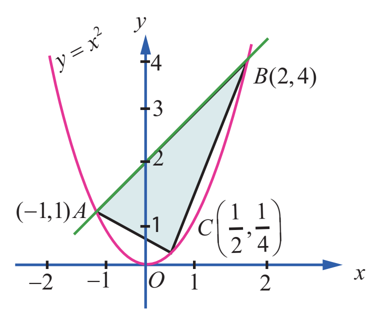


Fig. 9.1

Learning Objectives

Upon completion of this Chapter, students will be able to

- define a definite integral as the limit of a sum
- demonstrate a definite integral geometrically
- use the fundamental theorem of integral calculus
- evaluate definite integrals by evaluating anti-derivatives
- establish some properties of definite integrals
- identify improper integrals and use the gamma integral
- derive reduction formulae
- apply definite integral to evaluate area of a plane region
- apply definite integral to evaluate the volume of a solid of revolution

We briefly recall what we have already studied about anti-derivative of a given function $f(x)$. If a function $F(x)$ can be found such that $\frac{d}{dx}F(x) = f(x)$, then the function $F(x)$ is called an **anti-derivative** of $f(x)$.

It is not unique, because, for any arbitrary constant C, we get $\frac{d}{dx}[F(x)+C]=\frac{d}{dx}[F(x)]=f(x)$.

That is, if $F(x)$ is an anti-derivative of $f(x)$, then the function $F(x)+C$ is also an anti-derivative of the same function $f(x)$. Note that all anti-derivatives of $f(x)$ differ by a constant only. The anti-derivative of $f(x)$ is usually called the **indefinite integral** of $f(x)$ with respect to x and is denoted by $\int f(x)dx$.

A well-known property of indefinite integral is its **linear property** :

$\int[\alpha f(x)+\beta g(x)]dx=\alpha\int f(x)dx+\beta\int g(x)dx$, where α and β are constants.

We list below some functions and their anti-derivatives (indefinite integrals):

Function $f(x)$	**Indefinite integral** $\int f(x)dx$
K, a constant	$Kx+C$
$(ax+b)^n$, where $a\neq 0$ and b are constants; and $n\neq -1$	$\frac{1}{a}\left[\frac{(ax+b)^{n+1}}{n+1}\right]+C$
$\frac{1}{ax+b}$, where $a\neq 0$ and b are constants	$\frac{1}{a}\log_e\lvert(ax+b)\rvert+C$
e^{ax}, where a is a non-zero constant	$\frac{e^{ax}}{a}+C$
$\sin(ax+b)$, where $a\neq 0$ and b are constants	$-\frac{\cos(ax+b)}{a}+C$
$\cos(ax+b)$, where $a\neq 0$ and b are constants	$\frac{\sin(ax+b)}{a}+C$
$\tan(ax+b)$, where $a\neq 0$ and b are constants	$\frac{1}{a}\log\lvert\sec(ax+b)\rvert+C$
$\cot(ax+b)$, where $a\neq 0$ and b are constants	$\frac{1}{a}\log\lvert\sin(ax+b)\rvert+C$
$\sec(ax+b)$, where $a\neq 0$ and b are constants	$\frac{1}{a}\log\lvert\sec(ax+b)+\tan(ax+b)\rvert+C$
$\text{cosec}(ax+b)$, where $a\neq 0$ and b are constants	$-\frac{1}{a}\log\lvert\text{cosec}(ax+b)-\cot(ax+b)\rvert+C$
$\frac{1}{a^2+x^2}$, where $a\neq 0$ is a constant	$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$
$\frac{1}{a^2-x^2}$, where $a\neq 0$ is a constant	$\frac{1}{2a}\log_e\left\lvert\frac{a+x}{a-x}\right\rvert+C$
$\frac{1}{x^2-a^2}$, where $a\neq 0$ is a constant	$\frac{1}{2a}\log_e\left\lvert\frac{x-a}{x+a}\right\rvert+C$

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Function $f(x)$	Indefinite integral $\int f(x)dx$
$\dfrac{1}{\sqrt{a^2+x^2}}$, where a is a constant	$\log_e\left\|x+\sqrt{a^2+x^2}\right\|+C$
$\dfrac{1}{\sqrt{a^2-x^2}}$, where $a\neq 0$ is a constant	$\sin^{-1}\left(\dfrac{x}{a}\right)+C$
$\dfrac{1}{\sqrt{x^2-a^2}}$, where a is a constant	$\log_e\left\|x+\sqrt{x^2-a^2}\right\|+C$
$\sqrt{a^2+x^2}$, where a is a constant	$\dfrac{x\sqrt{a^2+x^2}}{2}+\dfrac{a^2}{2}\log_e\left\|x+\sqrt{a^2+x^2}\right\|+C$
$\sqrt{a^2-x^2}$, where a is a constant	$\dfrac{x\sqrt{a^2-x^2}}{2}+\dfrac{a^2}{2}\sin^{-1}\left(\dfrac{x}{a}\right)+C$
$\sqrt{x^2-a^2}$, where a is a constant	$\dfrac{x\sqrt{x^2-a^2}}{2}-\dfrac{a^2}{2}\log_e\left\|x+\sqrt{x^2-a^2}\right\|+C$

9.2 Definite Integral as the Limit of a Sum

9.2.1 Riemann Integral

Consider a real-valued, bounded function $f(x)$ defined on the closed and bounded interval $[a,b]$, $a<b$. The function $f(x)$ need not have the same sign on $[a,b]$; that is, $f(x)$ may have positive as well as negative values on $[a,b]$. See Fig 9.2. Partition the interval $[a,b]$ into n subintervals $[x_0,x_1],[x_1,x_2],\cdots,[x_{n-2},x_{n-1}],[x_{n-1},x_n]$ such that

$$a=x_0<x_1<x_2<\cdots<x_{n-1}<x_n=b.$$

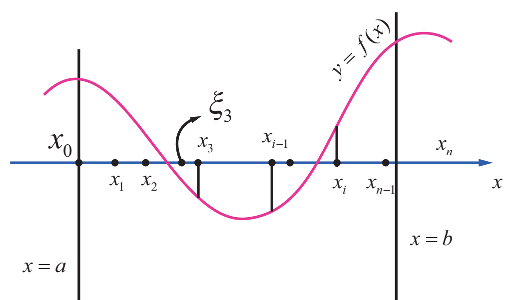


Fig. 9.2

In each subinterval $[x_{i-1},x_i], i=1,2,\cdots,n,$ choose a real number ξ_i arbitrarily such that $x_{i-1}\le\xi_i\le x_i$.

Consider the sum $\sum_{i=1}^{n} f(\xi_i)(x_i-x_{i-1}) = f(\xi_1)(x_1-x_0)+f(\xi_2)(x_2-x_1)+\cdots+f(\xi_n)(x_n-x_{n-1})$ ....(1)

The sum in (1) is called a **Riemann sum** of $f(x)$ corresponding to the partition $[x_0,x_1],[x_1,x_2],\cdots,[x_{n-1},x_n]$ of $[a,b]$. Since there are infinitely many values ξ_i satisfying the condition $x_{i-1}\le\xi_i\le x_i$, there are infinitely many Riemann sums of $f(x)$ corresponding to the same partition $[x_0,x_1],[x_1,x_2],\cdots,[x_{n-1},x_n]$ of $[a,b]$. If, under the limiting process $n\to\infty$ and $\max\left(x_i-x_{i-1}\right)\to 0$, the sum in (1) tends to a finite value, say A, then the value A is called the **definite integral** of $f(x)$ with respect to x on $[a,b]$. It is also called the **Riemann integral** of $f(x)$ on $[a,b]$ and is denoted by $\int_a^b f(x)dx$ and is read as the integral of $f(x)$ with respect to x from a to b. If $a=b$, then we have

$$\int_a^a f(x)dx=0.$$

Note

In the present chapter, we consider bounded functions $f(x)$ that are continuous on $[a,b]$. However, the **Riemann integral** of $f(x)$ on $[a,b]$ also exists for bounded functions $f(x)$ that are piece-wise continuous on $[a,b]$. We have used the same symbol $\int$ both for definite integral and anti-derivative (indefinite integral). The reason will be clear after we state the Fundamental Theorems of Integral Calculus. The variable x is **dummy** in the sense that it is selected at our choice only. So we can write $\int_a^b f(x)dx$ as $\int_a^b f(u)du$. So, we have $\int_a^b f(x)dx = \int_a^b f(u)du$. As $\max\left(x_i - x_{i-1}\right) \to 0$, all the three points x_{i-1}, ξ_i, and x_i of each subinterval $[x_{i-1}, x_i]$ are dragged into a single point. We have already indicated that there are infinitely many ways of choosing the evaluation point ξ_i in the subinterval $[x_{i-1}, x_i]$, $i = 1, 2, \cdots, n$. By choosing $\xi_i = x_{i-1}$, $i = 1, 2, \cdots, n$, we have

$$\int_a^b f(x)dx = \lim_{n\to\infty \text{ and } \max(x_i - x_{i-1})\to 0} \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1}). \qquad ...(2)$$

Equation (2) is known as the **left-end rule** for evaluating the Riemann integral.

By choosing $\xi_i = x_i$, $i = 1, 2, \cdots, n$, we have

$$\int_a^b f(x)dx = \lim_{n\to\infty \text{ and } \max(x_i - x_{i-1})\to 0} \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}). \qquad ...(3)$$

Equation (3) is known as the **right-end rule** for evaluating the Riemann integral.

By choosing $\xi_i = \dfrac{x_{i-1} + x_i}{2}$, $i = 1, 2, \cdots, n$, we have

$$\int_a^b f(x)dx = \lim_{n\to\infty \text{ and } \max(x_i - x_{i-1})\to 0} \sum_{i=1}^{n} f\left(\frac{x_{i-1} + x_i}{2}\right)(x_i - x_{i-1}). \qquad ...(4)$$

Equation (4) is known as the **mid-point rule** for evaluating the Riemann integral.

Remarks

(1) If the Riemann integral $\int_a^b f(x)dx$ exists, then the Riemann integral $\int_a^x f(u)du$ is a well-defined real number for every $x \in [a,b]$. So, we can define a function $F(x)$ on $[a,b]$ such that $F(x) = \int_a^x f(u)du, x \in [a,b]$.

(2) If $f(x) \geq 0$ for all $x \in [a,b]$, then the Riemann integral $\int_a^b f(x)dx$ is equal to the geometric area of the region bounded by the graph of $y = f(x)$, the x-axis, the lines $x = a$ and $x = b$. See Fig. 9.3.

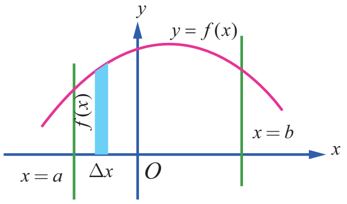


Fig. 9.3

(3) If $f(x) \leq 0$ for all $x \in [a,b]$, then the Riemann integral $\int_a^b f(x)dx$ is equal to the negative of the geometric area of the region bounded by the graph of $y = f(x)$, the x-axis, the lines $x = a$ and $x = b$. See Fig. 9.4. In this case, the geometric area of the region bounded by the graph of $y = f(x)$, the x-axis, the lines $x = a$ and $x = b$ is given by $\left|\int_a^b f(x)dx\right|$.

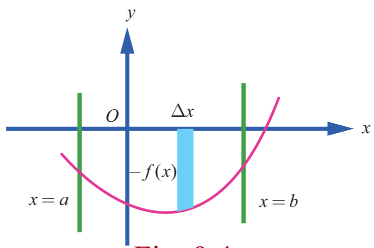


Fig. 9.4

(4) If $f(x)$ takes positive as well as negative values on $[a,b]$, then the interval $[a,b]$ can be divided into subintervals $[a,c_1]$, $[c_1,c_2]$, $\cdots$, $[c_k,b]$ such that $f(x)$ has the same sign throughout each of subintervals. So, the Riemann integral $\int_a^b f(x)dx$ is given by

$$\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \cdots + \int_{c_k}^{b} f(x)dx.$$

In this case, the geometric area of the region bounded by the graph of $y = f(x)$, the x-axis, the lines $x = a$ and $x = b$ is given by

$$\left|\int_a^{c_1} f(x)dx\right| + \left|\int_{c_1}^{c_2} f(x)dx\right| + \cdots + \left|\int_{c_k}^{b} f(x)dx\right|.$$

For instance, consider the following graph of a function $f(x), x \in [a,b]$. See Fig. 9.5. Here, A_1, A_2 and, A_3 denote geometric areas of the individual parts. Then, the definite integral $\int_a^b f(x)dx$ is given by

$$\begin{aligned}\int_a^b f(x)dx &= \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \int_{c_2}^{b} f(x)dx \\ &= A_1 - A_2 + A_3.\end{aligned}$$

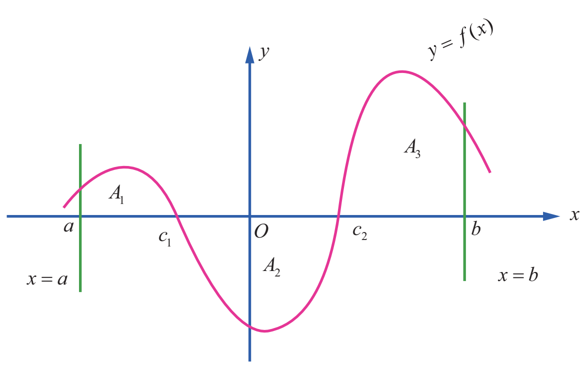


Fig. 9.5

The geometric area of the region bounded by the graph of $y = f(x)$, the $x-$axis, the lines $x = a$ and $x = b$ is given by $A_1 + A_2 + A_3$. In view of the above discussion, it is clear that a Riemann integral need not represent geometrical area.

Note

Even if we do not mention explicitly, it is always understood that the areas are measured in square units and volumes are measured in cubic units.

Example 9.1

Estimate the value of $\int_0^{0.5} x^2 dx$ using the Riemann sums corresponding to 5 subintervals of equal width and applying (i) left-end rule (ii) right-end rule (iii) the mid-point rule.

Solution

$$\text{Here } a = 0, b = 0.5, n = 5, f(x) = x^2$$

So, the width of each subinterval is

$$h = \Delta x = \frac{b-a}{n} = \frac{0.5-0}{5} = 0.1.$$

The partition of the interval is given by the points

$$\begin{aligned} x_0 &= 0, \\ x_1 &= x_0 + h = 0 + 0.1 = 0.1 \\ x_2 &= x_1 + h = 0.1 + 0.1 = 0.2 \\ x_3 &= x_2 + h = 0.2 + 0.1 = 0.3 \\ x_4 &= x_3 + h = 0.3 + 0.1 = 0.4 \\ x_5 &= x_4 + h = 0.4 + 0.1 = 0.5 \end{aligned}$$

(i) The left-end rule for Riemann sum with equal width Δx is

$$\begin{aligned} S &= \left[f(x_0) + f(x_1) + \cdots + f(x_{n-1})\right]\Delta x. \\ \therefore S &= \left[f(0) + f(0.1) + f(0.2) + f(0.3) + f(0.4)\right](0.1) \\ &= [0.00 + 0.01 + 0.04 + 0.09 + 0.16](0.1) = 0.03 \end{aligned}$$

$\therefore \int_0^{0.5} x^2 dx$ is approximately 0.03.

(ii) The right-end rule for Riemann sum with equal width Δx is

$$\begin{aligned} S &= \left[f(x_1) + f(x_2) + \cdots + f(x_n)\right]\Delta x. \\ \therefore S &= \left[f(0.1) + f(0.2) + f(0.3) + f(0.4) + f(0.5)\right](0.1) \\ &= [0.01 + 0.04 + 0.09 + 0.16 + 0.25](0.1) = 0.055. \end{aligned}$$

$\therefore \int_0^{0.5} x^2 dx$ is approximately 0.055.

(iii)The mid-point rule for Riemann sum with equal width Δx is

$$\begin{aligned} S &= \left[f\left(\frac{x_0+x_1}{2}\right) + f\left(\frac{x_1+x_2}{2}\right) + \cdots + f\left(\frac{x_{n-1}+x_n}{2}\right)\right]\Delta x \\ \therefore S &= \left[f(0.05) + f(0.15) + f(0.25) + f(0.35) + f(0.45)\right](0.1) \\ &= [0.0025 + 0.0225 + 0.0625 + 0.1225 + 0.2025](0.1) \\ &= 0.04125. \end{aligned}$$

$\therefore \int_0^{0.5} x^2 dx$ is approximately 0.04125.

EXERCISE 9.1

1. Find an approximate value of $\int_1^{1.5} x dx$ by applying the left-end rule with the partition $\{1.1, 1.2, 1.3, 1.4, 1.5\}$.

2. Find an approximate value of $\int_1^{1.5} x^2 dx$ by applying the right-end rule with the partition $\{1.1, 1.2, 1.3, 1.4, 1.5\}$.

3. Find an approximate value of $\int_1^{1.5} (2-x) dx$ by applying the mid-point rule with the partition $\{1.1, 1.2, 1.3, 1.4, 1.5\}$.

9.2.2 Limit Formula to Evaluate $\int_a^b f(x)dx$

Divide the interval $[a,b]$ into n equal subintervals $[x_0,x_1],[x_1,x_2],\cdots,[x_{n-2},x_{n-1}],[x_{n-1},x_n]$ such that $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$. Then, we have $x_1 - x_0 = x_2 - x_1 = \cdots = x_n - x_{n-1} = \frac{b-a}{n}$. Put $h = \frac{b-a}{n}$. Then, we get $x_i = a + ih, i = 1,2,\cdots,n.$

So, by the definition of definite integral, we get

$$\lim_{n\to\infty \text{ and } \max(x_i - x_{i-1})\to 0} \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) \text{ (Right-end rule)}$$

$$= \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right).$$

$$\therefore \quad \int_a^b f(x)dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{r=1}^{n} f\left(a + (b-a)\frac{r}{n}\right).$$

Note. $\lim_{n\to\infty} \frac{b-a}{n} \sum_{r=0}^{n} f\left(a + (b-a)\frac{r}{n}\right) = \lim_{n\to\infty} \left[\frac{b-a}{n} f(a) + \frac{b-a}{n} \sum_{r=1}^{n} f\left(a + (b-a)\frac{r}{n}\right)\right]$

$$= \lim_{n\to\infty} \frac{b-a}{n} \sum_{r=1}^{n} f\left(a + (b-a)\frac{r}{n}\right)$$

$$= \int_a^b f(x)dx.$$

$$\therefore \int_a^b f(x)dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{r=1}^{n} f\left(a + (b-a)\frac{r}{n}\right)$$

$$= \lim_{n\to\infty} \frac{b-a}{n} \sum_{r=0}^{n} f\left(a + (b-a)\frac{r}{n}\right).$$

If $a = 0$ and $b = 1$, then we get $\int_0^1 f(x)dx = \lim_{n\to\infty} \frac{1}{n} \sum_{r=0}^{n} f\left(\frac{r}{n}\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} f\left(\frac{r}{n}\right).$

Example 9.2

Evaluate $\int_0^1 x\,dx$, as the limit of a sum.

Solution

Here $f(x)=x,\ a=0$ and $b=1$. Hence, we get

$$\int_a^b f(x)dx = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right) \Rightarrow \int_0^1 x\,dx = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{r}{n}$$

$$= \lim_{n\to\infty}\frac{1}{n^2}\left[1+2+\cdots+n\right]$$

$$= \lim_{n\to\infty}\frac{1}{n^2}\frac{n(n+1)}{2} = \lim_{n\to\infty}\frac{1}{2}\left(1+\frac{1}{n}\right) = \frac{1}{2}.$$

Example 9.3

Evaluate $\int_0^1 x^3 dx$, as the limit of a sum.

Solution

Here $f(x)=x^3,\ a=0$ and $b=1$. Hence, we get

$$\int_a^b f(x)dx = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right) \Rightarrow \int_0^1 x^3 dx = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{r^3}{n^3}$$

$$= \lim_{n\to\infty}\frac{1}{n^4}\left[1^3+2^3+\cdots+n^3\right] = \lim_{n\to\infty}\frac{1}{n^4}\frac{n^2(n+1)^2}{4}$$

$$= \lim_{n\to\infty}\frac{1}{4}\left(1+\frac{1}{n}\right)^2 = \frac{1}{4}.$$

Example 9.4

Evaluate $\int_1^4 \left(2x^2+3\right)dx$, as the limit of a sum.

Solution

We use the formula

$$\int_a^b f(x)dx = \lim_{n\to\infty}\frac{b-a}{n}\sum_{r=1}^{n} f\left(a+(b-a)\frac{r}{n}\right)$$

Here $f(x)=2x^2+3,\ a=1$ and $b=4$.

So, we get

$$f\left(a+(b-a)\frac{r}{n}\right) = f\left(1+(4-1)\frac{r}{n}\right) = f\left(1+\frac{3r}{n}\right) = 2\left(1+\frac{3r}{n}\right)^2+3 = 5+\frac{18r^2}{n^2}+\frac{12r}{n}.$$

Hence, we get

$$\int_1^4 (2x^2+3)dx = \lim_{n\to\infty}\frac{3}{n}\sum_{r=1}^{n}\left(5+\frac{18r^2}{n^2}+\frac{12r}{n}\right) = \lim_{n\to\infty}\left[\frac{15}{n}\sum_{r=1}^{n}1+\frac{54}{n^3}\sum_{r=1}^{n}r^2+\frac{36}{n^2}\sum_{r=1}^{n}r\right]$$

$$= \lim_{n\to\infty}\left[\frac{15}{n}n+\frac{54}{n^3}\left(1^2+2^2+\cdots+n^2\right)+\frac{36}{n^2}\left(1+2+\cdots+n\right)\right]$$

$$= \lim_{n\to\infty}\left[15+\frac{54}{n^3}\frac{n(n+1)(2n+1)}{6}+\frac{36}{n^2}\frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty}\left[15+9\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+18\left(1+\frac{1}{n}\right)\right]$$

$$= 15+9\left(1+0\right)\left(2+0\right)+18\left(1+0\right)=51.$$

EXERCISE 9.2

N2A4Q9

1. Evaluate the following integrals as the limits of sums:

(i) $\int_0^1 (5x+4)dx$ (ii) $\int_1^2 (4x^2-1)dx$

9.3 Fundamental Theorems of Integral Calculus and their Applications

We observe in the above examples that evaluation of $\int_a^b f(x)dx$ as a limit of the sum is quite tedious, even if $f(x)$ is a very simple function. Both Newton and Leibnitz, more or less at the same time, devised an easy method for evaluating definite integrals. Their method is based upon two celebrated theorems known as **First Fundamental Theorem and Second Fundamental Theorem of Integral Calculus.** These theorems establish the connection between a function and its anti-derivative (if it exists). In fact, the two theorems provide a link between differential calculus and integral calculus. We state below the above important theorems without proofs.

Theorem 9.1 (First Fundamental Theorem of Integral Calculus)

If $f(x)$ be a continuous function defined on a closed interval $[a,b]$ and $F(x)=\int_a^x f(u)du,\;\; a<x<b$

then, $\frac{d}{dx}F(x)=f(x)$. In other words, $F(x)$ is an anti-derivative of $f(x)$.

Theorem 9.2 (Second Fundamental Theorem of Integral Calculus)

If $f(x)$ be a continuous function defined on a closed interval $[a,b]$ and $F(x)$ is an anti-derivative of $f(x)$, then,

$$\int_a^b f(x)dx = F(b)-F(a).$$

Note

Since $F(b)-F(a)$ is the value of the definite integral (Riemann integral) $\int_a^b f(x)dx$, any arbitrary constant added to the anti-derivative $F(x)$ cancels out and hence it is not necessary to add an arbitrary constant to the anti-derivative, when we are evaluating definite integrals. As a short-hand form, we write $F(b)-F(a)=\left[F(x)\right]_a^b$. The value of a definite integral is unique.

By the second fundamental theorem of integral calculus, the following properties of definite integrals hold. They are stated here without proof.

Property 1 : $\int\limits_a^b f(x)\,dx = \int\limits_a^b f(u)\,du$, $a < b$

i.e., definite integral is independent of the change of variable.

Property 2 : $\int\limits_a^b f(x)dx = -\int\limits_b^a f(x)dx$

i.e., the value of the definite integral changes by minus sign if the limits are interchanged.

Property 3 : $\int\limits_a^b f(x)dx = \int\limits_a^c f(x)dx + \int\limits_c^b f(x)dx$, $a < c < b$

Property 4 : $\int_a^b [\alpha f(x) + \beta g(x)]dx = \alpha\int_a^b f(x)dx + \beta\int_a^b g(x)dx$, where α and β are constants.

Property 5 : If $x = g(u)$, then $\int_a^b f(x)dx = \int_c^d f(g(u))\dfrac{dg(u)}{du}du$ where $g(c) = a$ and $g(d) = b$.

This property is used for evaluating definite integrals by making substitution.

We illustrate the use of the above properties by the following examples.

Example 9.5

Evaluate : $\int\limits_0^3 (3x^2 - 4x + 5)dx$.

Solution

$$\begin{aligned}\int\limits_0^3 (3x^2 - 4x + 5)dx &= \int\limits_0^3 3x^2 dx - \int\limits_0^3 4x\ dx + \int\limits_0^3 5\,dx \\ &= 3\int\limits_0^3 x^2 dx - 4\int\limits_0^3 x\,dx + 5\int\limits_0^3 dx \\ &= 3\left[\frac{x^3}{3}\right]_0^3 - 4\left[\frac{x^2}{2}\right]_0^3 + 5[x]_0^3 \\ &= (27-0) - 2(9-0) + 5(3-0) \\ &= 27 - 18 + 15 = 24 .\end{aligned}$$

Example 9.6

Evaluate : $\int\limits_0^1 \dfrac{2x+7}{5x^2+9}dx$.

Solution

$$\int\limits_0^1 \frac{2x+7}{5x^2+9}dx = \int\limits_0^1 \frac{2x}{5x^2+9} + 7\int\limits_0^1 \frac{dx}{(5x^2)+3^2} = \frac{1}{5}\log[5x^2+9]_0^1 + \frac{7}{5}\int\limits_0^1 \frac{dx}{x^2 + \left(\frac{3}{\sqrt{5}}\right)^2}$$

$$= \frac{1}{5}[\log 14 - \log 9] + \frac{7}{5} \times \frac{\sqrt{5}}{3}\left[\tan^{-1}\frac{x}{\left[\frac{3}{\sqrt{5}}\right]}\right]_0^1 = \frac{1}{5}\log\frac{14}{9} + \frac{7}{3\sqrt{5}}\tan^{-1}\frac{\sqrt{5}}{3}.$$

Example 9.7

Evaluate : $\int_0^1 [2x]\,dx$ where $[\cdot]$ is the greatest integer function.

Solution

$$\int_0^1 [2x]\,dx = \int_0^{\frac{1}{2}} [2x]\,dx + \int_{\frac{1}{2}}^1 [2x]\,dx = \int_0^{\frac{1}{2}} 0\,dx + \int_{\frac{1}{2}}^1 1\,dx = 0 + [x]_{\frac{1}{2}}^1 = 1 - \frac{1}{2} = \frac{1}{2}.$$

Example 9.8

Evaluate : $\int_0^{\frac{\pi}{3}} \frac{\sec x \tan x}{1+\sec^2 x}\,dx$.

Solution

Let $I = \int_0^{\frac{\pi}{3}} \frac{\sec x \tan x}{1+\sec^2 x}\,dx$. Put $\sec x = u$. Then, $\sec x \tan x\; dx = du$.

When $x = 0$, $u = \sec 0 = 1$. When $x = \frac{\pi}{3}$, $u = \sec\frac{\pi}{3} = 2$.

$$\therefore I = \int_1^2 \frac{du}{1+u^2} = [\tan^{-1} u]_1^2 = \tan^{-1}(2) - \tan^{-1} 1 = \tan^{-1}(2) - \frac{\pi}{4}.$$

Example 9.9

Evaluate : $\int_0^9 \frac{1}{x+\sqrt{x}}\,dx$.

Solution

Let $\sqrt{x} = u$. Then $x = u^2$, and so $dx = 2u\,du$.

When $x = 0$, $u = 0$. When $x = 9$, $u = 3$.

$$\therefore \int_0^9 \frac{1}{x+\sqrt{x}}\,dx = \int_0^3 \frac{1}{u^2+u}(2u)\,du = 2\int_0^3 \frac{1}{1+u}\,du = 2\left[\log|1+u|\right]_0^3 = 2[\log 4 - 0] = \log 16.$$

Example 9.10

Evaluate: $\int_1^2 \frac{x}{(x+1)(x+2)}\,dx$.

Solution

$$\text{Let } I = \int_1^2 \frac{x}{(x+1)(x+2)}\,dx.$$

$$I = \int_1^2 \left[\frac{-1}{(x+1)} + \frac{2}{x+2}\right] dx \qquad \text{(Using partial fractions)}$$

$$= \left[-\log(x+1) + 2\log(x+2)\right]_1^2$$

$$= \log\left[\frac{(x+2)^2}{x+1}\right]_1^2$$

$$= \log\frac{16}{3} - \log\frac{9}{2}$$

$$= \log\frac{32}{27}.$$

Example 9.11

Evaluate : $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos\theta}{(1+\sin\theta)(2+\sin\theta)} d\theta$.

Solution

Let $I = \displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos\theta}{(1+\sin\theta)(2+\sin\theta)} d\theta$. Put $u = 1+\sin\theta$. Then, $du = \cos\theta\, d\theta$.

When $\theta = 0$, $u = 1$. When $\theta = \dfrac{\pi}{2}$, $u = 2$.

$$\therefore I = \int_1^2 \frac{du}{u(1+u)} = \int_1^2 \frac{(1+u)-u}{u(1+u)} du = \int_1^2 \left(\frac{1}{u} - \frac{1}{1+u}\right) du = \left[\log u - \log(1+u)\right]_1^2$$

$$= (\log 2 - \log 3) - (\log 1 - \log 2) = 2\log 2 - \log 3 = \log\frac{4}{3}.$$

Example 9.12

Evaluate : $\displaystyle\int_0^{\frac{1}{\sqrt{2}}} \frac{\sin^{-1} x}{(1-x^2)^{\frac{3}{2}}} dx$.

Solution

Let $I = \displaystyle\int_0^{\frac{1}{\sqrt{2}}} \frac{\sin^{-1} x}{(1-x^2)^{\frac{3}{2}}} dx$.

Put $u = \sin^{-1} x$. Then, $x = \sin u$ and so, $du = \dfrac{1}{\sqrt{1-x^2}} dx$.

When $x = 0$, $u = 0$. When $x = \dfrac{1}{\sqrt{2}}$, $u = \dfrac{\pi}{4}$.

$$\therefore I = \int_0^{\frac{\pi}{4}} \frac{u}{\cos^2 u} du = \int_0^{\frac{\pi}{4}} u \sec^2 u \, du = \left[u \tan u\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan u \, du = \left[u \tan u\right]_0^{\frac{\pi}{4}} + \left[\log\cos u\right]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{4} + \log\frac{1}{\sqrt{2}} = \frac{\pi}{4} - \frac{1}{2}\log 2.$$

Example 9.13

Evaluate : $\int_0^{\frac{\pi}{2}} \left(\sqrt{\tan x} + \sqrt{\cot x}\right) dx$.

Solution

Let $I = \int_0^{\frac{\pi}{2}} \left(\sqrt{\tan x} + \sqrt{\cot x}\right) dx$. Then, we get

$$I = \int_0^{\frac{\pi}{2}} \left(\sqrt{\frac{\sin x}{\cos x}} + \sqrt{\frac{\cos x}{\sin x}}\right) dx = \int_0^{\frac{\pi}{2}} \frac{\sin x + \cos x}{\sqrt{\sin x \cos x}} dx = \sqrt{2} \int_0^{\frac{\pi}{2}} \frac{\sin x + \cos x}{\sqrt{2 \sin x \cos x}} dx$$

$$= \sqrt{2} \int_0^{\frac{\pi}{2}} \frac{(\sin x + \cos x) dx}{\sqrt{1 - (\sin x - \cos x)^2}}.$$

Put $u = \sin x - \cos x$. Then, $du = (\cos x + \sin x) dx$.

When $x = 0$, $u = -1$. When $x = \frac{\pi}{2}$, $u = 1$.

$$\therefore I = \sqrt{2} \int_{-1}^{1} \frac{du}{\sqrt{1-u^2}} = \sqrt{2}[\sin^{-1} u]_{-1}^{1} = \sqrt{2}\left[\sin^{-1}(1) - \sin^{-1}(-1)\right] = \pi\sqrt{2} \ .$$

Example 9.14

Evaluate : $\int_0^{1.5} \left[x^2\right] dx$, where $[x]$ is the greatest integer function.

Solution

We know that the greatest integer function $[x]$ is the largest integer less than or equal to x. In other words, it is defined by $[x] = n$, if $n \le x < (n+1)$, where n is an integer.

So, we get $\quad [x^2] = \begin{cases} 0 & \text{if} \quad 0 \le x < 1 \\ 1 & \text{if} \quad 1 \le x < \sqrt{2} \\ 2 & \text{if} \quad \sqrt{2} \le x \le 1.5 \end{cases}$

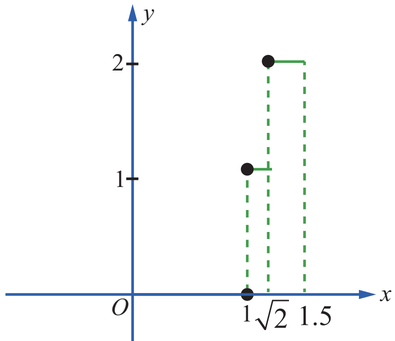


Fig. 9.6

We note that the above function is not continuous on $[0,1.5]$.

But, it is continuous in each of the sub-intervals $[0,1)$, $[1,\sqrt{2})$ and $[\sqrt{2},1.5]$; that is, it is piece-wise continuous on $[0,1.5]$. See Fig. 9.6. Hence, we get

$$\int_0^{1.5} \left[x^2\right] dx = \int_0^{1} \left[x^2\right] dx + \int_1^{\sqrt{2}} \left[x^2\right] dx + \int_{\sqrt{2}}^{1.5} \left[x^2\right] dx = \int_0^1 0\, dx + \int_1^{\sqrt{2}} 1\, dx + \int_{\sqrt{2}}^{1.5} 2 dx$$

$$= 0 + (x)_1^{\sqrt{2}} + (2x)_{\sqrt{2}}^{1.5} = \left(\sqrt{2} - 1\right) + \left(3 - 2\sqrt{2}\right) = 2 - \sqrt{2}.$$

Example 9.15

Evaluate : $\int_{-4}^{4} | x + 3 | \ dx$.

Solution

By definition, we have $|x+3| = \begin{cases} x+3 & \text{if } x \ge -3 \\ -x-3 & \text{if } x < -3 \end{cases}$

See Fig. 9.7 for the graph of $y = |x+3|$ in $-4 \le x \le 4$.

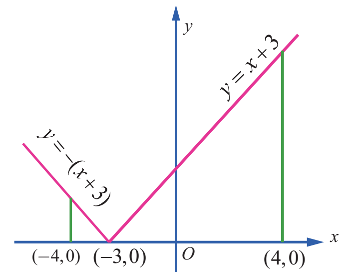


Fig. 9.7

$$\therefore \int_{-4}^{4} |x+3|\, dx = \int_{-4}^{-3} |x+3|\, dx + \int_{-3}^{4} |x+3|\, dx = \int_{-4}^{-3} (-x-3)\, dx + \int_{-3}^{4} (x+3)\, dx$$

$$= \left[-\frac{x^2}{2} - 3x\right]_{-4}^{-3} + \left[\frac{x^2}{2} + 3x\right]_{-3}^{4}$$

$$= \left(-\frac{9}{2}+9\right) - \left(-\frac{16}{2}+12\right) + \left(\frac{16}{2}+12\right) - \left(\frac{9}{2}-9\right) = \left(\frac{9}{2}\right) - 4 + 20 + \left(\frac{9}{2}\right) = 25.$$

Next, we give examples to illustrate the application of Property 5.

Example 9.16

Show that $\int_0^{\frac{\pi}{2}} \frac{dx}{4+5\sin x} = \frac{1}{3}\log_e 2$.

Solution

Put $u = \tan\frac{x}{2}$. Then, $\sin x = \dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \dfrac{2u}{1+u^2}$, $du = \dfrac{1}{2}\sec^2\dfrac{x}{2}\, dx \Rightarrow dx = \dfrac{2du}{1+u^2}$.

When $x = 0, u = \tan 0 = 0$. When $x = \frac{\pi}{2}, u = \tan\frac{\pi}{4} = 1$.

$$\therefore I = \int_0^{\frac{\pi}{2}} \frac{dx}{4+5\sin x} = \int_0^1 \frac{\frac{2du}{1+u^2}}{4+5\left(\frac{2u}{1+u^2}\right)} = \int_0^1 \frac{du}{2u^2+5u+2} = \frac{1}{2}\int_0^1 \frac{du}{u^2+\frac{5}{2}u+1}$$

$$= \frac{1}{2}\int_0^1 \frac{du}{\left(u+\frac{5}{4}\right)^2 - \left(\frac{3}{4}\right)^2} = \left[\frac{1}{2} \times \frac{1}{2\times\left(\frac{3}{4}\right)} \log\left(\frac{\left(u+\frac{5}{4}\right)-\frac{3}{4}}{\left(u+\frac{5}{4}\right)+\frac{3}{4}}\right)\right]_0^1 = \frac{1}{3}\left[\log\left(\frac{u+\frac{1}{2}}{u+2}\right)\right]_0^1 = \frac{1}{3}\log 2.$$

Note

To evaluate anti-derivatives of the type $\int \frac{dx}{a\cos x + b\sin x + c}$, we use the substitution method by putting $u = \tan\frac{x}{2}$ so that $\cos x = \frac{1-u^2}{1+u^2}, \sin x = \frac{2u}{1+u^2}, dx = \frac{2du}{1+u^2}$.

Example 9.17

Prove that $\int_0^{\frac{\pi}{4}} \frac{\sin 2x\, dx}{\sin^4 x + \cos^4 x} = \frac{\pi}{4}$.

Solution

$$I = \int_0^{\frac{\pi}{4}} \frac{\sin 2x\, dx}{\sin^4 x + \cos^4 x} = \int_0^{\frac{\pi}{4}} \frac{\sin 2x\, dx}{\left(\sin^2 x + \cos^2 x\right)^2 - 2\sin^2 x \cos^2 x}$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sin 2x\, dx}{1 - \frac{1}{2}\left(2\sin x \cos x\right)^2} = \int_0^{\frac{\pi}{4}} \frac{2\sin 2x\, dx}{2 - \sin^2 2x} = \int_0^{\frac{\pi}{4}} \frac{2\sin 2x\, dx}{1 + \cos^2 2x}.$$

Put $u = \cos 2x$, Then, $du = -2\sin 2x\, dx$.

When $x = 0$, we have $u = \cos 0 = 1$. When $x = \frac{\pi}{4}$, we have $u = \cos\frac{\pi}{2} = 0$.

$$\therefore I = \int_1^0 \frac{-du}{1+u^2} = \int_0^1 \frac{du}{1+u^2} = \left[\tan^{-1} u\right]_0^1 = \frac{\pi}{4}.$$

■

Example 9.18

Prove that $\int_0^{\frac{\pi}{4}} \frac{dx}{a^2\sin^2 x + b^2\cos^2 x} = \frac{1}{ab}\tan^{-1}\left(\frac{a}{b}\right)$, where $a,\ b > 0$.

Solution

$$\text{Put } I = \int_0^{\frac{\pi}{4}} \frac{dx}{a^2\sin^2 x + b^2\cos^2 x} = \int_0^{\frac{\pi}{4}} \frac{\sec^2 x\, dx}{a^2\tan^2 x + b^2}.$$

Put $u = \tan x$. Then $du = \sec^2 x\, dx$.

When $x = 0$, we have $u = \tan 0 = 0$. When $x = \frac{\pi}{4}$, we have $u = \tan\frac{\pi}{4} = 1$.

$$\therefore I = \int_0^1 \frac{du}{a^2u^2 + b^2} = \frac{1}{a^2}\int_0^1 \frac{du}{u^2 + \left(\frac{b}{a}\right)^2} = \frac{1}{a^2}\left[\frac{a}{b}\tan^{-1}\left(\frac{au}{b}\right)\right]_0^1 = \frac{1}{ab}\tan^{-1}\left(\frac{a}{b}\right).$$

We derive some more properties of definite integrals. ■

Property 6

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

Proof

Let $u = a+b-x$. Then, we get $dx = -du$.

When $x = a$, $u = a+b-a = b$. When $x = b$, we get $u = a+b-b = a$.

$$\therefore \int_a^b f(x)dx = \int_b^a f(a+b-u)(-du) = \int_a^b f(a+b-u)du$$

$$= \int_a^b f(a+b-x)dx.$$

■

Note

Replace a by 0 and b by a in the above property we get the following property

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx.$$

Example 9.19

Evaluate $\int_0^{\frac{\pi}{4}} \frac{1}{\sin x + \cos x}\, dx$

Solution

$$I = \int_0^{\frac{\pi}{4}} \frac{1}{\sin x + \cos x}\, dx = \int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x\right)}\, dx$$

$$= \frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{4}} \frac{1}{\left(\cos\frac{\pi}{4}\cos x + \sin\frac{\pi}{4}\sin x\right)}\, dx = \frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{4}} \frac{1}{\cos\left(\frac{\pi}{4} - x\right)}\, dx$$

$$= \frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{4}} \frac{1}{\cos x}\, dx \text{, since } \int_0^a f(x)dx = \int_0^a f(a-x)dx$$

$$= \frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{4}} \sec x \; dx = \frac{1}{\sqrt{2}}\left[\log(\sec x + \tan x)\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{\sqrt{2}}\left[\log(\sqrt{2}+1) - \log(1+0)\right]$$

$$= \frac{1}{\sqrt{2}}\log(\sqrt{2}+1).$$

Property 7

$$\int_0^{2a} f(x)\,dx = \int_0^a \left[f(x) + f(2a-x)\right]dx.$$

Proof

By property 3, we have $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx$. (1)

Let us make the substitution $x = 2a-u$ in $\int_a^{2a} f(x)dx$. Then, $dx = -du$.

When $x = a$, we have $u = 2a - a = a$. When $x = 2a$, we have $u = 2a - 2a = 0$. So, we get

$$\int_a^{2a} f(x)dx = \int_a^0 f(2a-u)(-du) = \int_0^a f(2a-u)du = \int_0^a f(2a-x)dx. \quad \text{...(2)}$$

Substituting equation (2) in equation (1), we get

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$$

$$= \int_0^a \left[f(x) + f(2a-x)\right]dx.$$

Property 8

If $f(x)$ is an even function, then $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx.$

(Recall that a function $f(x)$ is an even function if and only if $f(-x) = f(x)$.)

Proof

By property 3, we have

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx.$$

In the integral $\int_{-a}^{0} f(x)\,dx$, let us make the substitution, $x=-u$. Then, $dx=-du$.

When $x = -a$, we get $u=a$, when $x=0$, we get $u=0$, So, we get

$$\int_{-a}^{0} f(x)\,dx = \int_{a}^{0} f(-u)(-du) = \int_{0}^{a} f(-u)\,du = \int_{0}^{a} f(-x)\,dx = \int_{0}^{a} f(x)\,dx. \quad \text{... (2)}$$

Substituting equation (2) in equation (1), we get

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

■

Property 9

If $f(x)$ is an odd function, then $\int_{-a}^{a} f(x)\,dx = 0.$

(Recall that a function $f(x)$ is an odd function if and only if $f(-x)=-f(x)$.)

Proof

By property 3, we have

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx. \quad \text{... (1)}$$

Consider $\int_{-a}^{0} f(x)\,dx$. In this integral, let us make the substitution, $x=-u$. Then, $dx=-du$.

When $x = -a$, we get $u=a$; when $x=0$, we get $u=0$. So, we get

$$\int_{-a}^{0} f(x)\,dx = \int_{a}^{0} f(-u)(-du) = \int_{0}^{a} f(-u)\,du = \int_{0}^{a} f(-x)\,dx = -\int_{0}^{a} f(x)\,dx. \quad \text{... (2)}$$

Substituting equation (2) in equation (1), we get

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(x)\,dx - \int_{0}^{a} f(x)\,dx = 0$$

■

Property 10

If $f(2a-x)=f(x)$, then $\int_{0}^{2a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$

Proof

By property 7, we have

$$\int_{0}^{2a} f(x)\,dx = \int_{0}^{a} \left[f(x)+f(2a-x)\right]dx. \quad \text{...(1)}$$

Setting the condition $f(2a-x)=f(x)$ in equation (1), we get

$$\int_{0}^{2a} f(x)\,dx = \int_{0}^{a} \left[f(x)+f(x)\right]dx = 2\int_{0}^{a} f(x)\,dx.$$

■

Property 11

If $f(2a-x)=-f(x)$, then $\int_{0}^{2a} f(x)\,dx = 0.$

Proof

By property 7, we have

$$\int_0^{2a} f(x)\,dx = \int_0^a [f(x)+f(2a-x)]\,dx. \qquad \text{... (1)}$$

Setting the condition $f(2a-x)=-f(x)$ in equation (1), we get

$$\int_0^{2a} f(x)\,dx = \int_0^a [f(x)-f(x)]\,dx = 0.$$

■

Property 12

$$\int_0^a x\,f(x)\,dx = \frac{a}{2}\int_0^a f(x)\,dx \text{ if } f(a-x)=f(x).$$

Proof

$$\text{Let } I = \int_0^a x\,f(x)\,dx \qquad \text{... (1)}$$

$$\text{Then } I = \int_0^a (a-x)f(a-x)\,dx, \text{ since } \int_0^a g(x)\,dx = \int_0^a g(a-x)\,dx$$

$$= \int_0^a (a-x)f(x)\,dx, \text{ since } f(a-x)=f(x).$$

$$\therefore I = \int_0^a (a-x)f(x)\,dx \qquad \text{... (2)}$$

Adding (1) and (2), we get

$$2I = \int_0^a (x+a-x)f(x)\,dx$$

$$= a\int_0^a f(x)\,dx.$$

$$\therefore I = \frac{a}{2}\int_0^a f(x)\,dx.$$

■

Note

This property help us to remove the factor x present in the integrand of the LHS.

Example 9.20

Show that $\int_0^{\pi} g(\sin x)dx = 2\int_0^{\frac{\pi}{2}} g(\sin x)dx$, where $g(\sin x)$ is a function of $\sin x$.

Solution

We know that

$$\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx \text{ if } f(2a-x)=f(x).$$

$$\text{Take } 2a=\pi \text{ and } f(x)=g(\sin x).$$

$$\text{Then, } f(2a-x) = g(\sin(\pi-x)) = g(\sin x) = f(x).$$

$$\therefore \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx.$$

$$\int_0^{\pi} g(\sin x)\,dx = 2\int_0^{\frac{\pi}{2}} g(\sin x)\,dx.$$

Result

$$\int_0^{\pi} g(\sin x)\,dx = 2\int_0^{\frac{\pi}{2}} g(\sin x)\,dx.$$

Note

The above result is useful in evaluating definite integrals of the type $\int_0^{\pi} g(\sin x)\,dx$.

Example 9.21

Evaluate $\int_0^{\pi} \frac{x}{1+\sin x}\,dx$.

Solution

$$\text{Let } I = \int_0^{\pi} \frac{x}{1+\sin x}\,dx.$$

$$= \int_0^{\pi} x \frac{1}{1+\sin x}\,dx$$

$$\text{Let } f(x) = \frac{1}{1+\sin x}. \text{ Then } f(\pi - x) = \frac{1}{1+\sin(\pi - x)} = \frac{1}{1+\sin x} = f(x)$$

$$\therefore \int_0^{\pi} \frac{x}{1+\sin x}\,dx = \frac{\pi}{2}\int_0^{\pi} \frac{1}{1+\sin x}\,dx, \quad \left(\because \int_0^{a} x f(x)\,dx = \frac{a}{2}\int_0^{a} f(x)\,dx \text{ if } f(a-x) = f(x)\right)$$

$$= \pi \int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x}\,dx, \quad \text{since } \int_0^{\pi} g(\sin x)\,dx = 2\int_0^{\frac{\pi}{2}} g(\sin x)\,dx$$

$$= \pi \int_0^{\frac{\pi}{2}} \frac{1}{1+\sin\left(\frac{\pi}{2} - x\right)}\,dx \quad \text{since } \int_0^{a} f(x)\,dx = \int_0^{a} f(a-x)\,dx$$

$$= \pi \int_0^{\frac{\pi}{2}} \frac{1}{1+\cos x}\,dx = \pi \int_0^{\frac{\pi}{2}} \frac{1}{2\cos^2 \frac{x}{2}}\,dx = \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \sec^2 \frac{x}{2}\,dx$$

$$= \pi \left[\tan \frac{x}{2}\right]_0^{\frac{\pi}{2}} = \pi \left[\tan \frac{\pi}{4} - \tan 0\right] = \pi.$$

Example 9.22

Show that $\int_0^{2\pi} g(\cos x)\,dx = 2\int_0^{\pi} g(\cos x)\,dx$, where $g(\cos x)$ is a function of $\cos x$.

Solution

$$\text{Take } 2a = 2\pi \text{ and } f(x) = g(\cos x).$$

Then, $f(2a-x) = f(2\pi - x) = g(\cos(2\pi - x)) = g(\cos x) = f(x)$

$$\therefore \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx.$$

$$\therefore \int_0^{2\pi} g(\cos x)\,dx = 2\int_0^{\pi} g(\cos x)\,dx.$$

Result

$$\int_0^{2\pi} g(\cos x)\,dx = 2\int_0^{\pi} g(\cos x)\,dx.$$

Note

The above result is useful in evaluating definite integrals of the type $\int_0^{2\pi} g(\cos x)\,dx$.

Example 9.23

If $f(x) = f(a+x)$, then $\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx$

Solution

We write $\int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_a^{2a} f(x)\,dx$ ... (1)

Consider $\int_a^{2a} f(x)\,dx$

Substituting $x = a+u$, we have $dx = du$; when $x = a, u = 0$ and when $x = 2a, u = a$.

$$\therefore \int_a^{2a} f(x)\,dx = \int_0^{a} f(a+u)\,du = \int_0^{a} f(u)\,du, \text{ since } f(x) = f(a+x)$$

$$= \int_0^{a} f(x)\,dx. \qquad \text{... (2)}$$

Substituting (2) in (1), we get

$$\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx.$$

Example 9.24

Evaluate : $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\cos x\,dx$.

Solution

Let $f(x) = x\cos x$. Then $f(-x) = (-x)\cos(-x) = -x\cos x = -f(x)$.

So $f(x) = x\cos x$ is an odd function.

Hence, applying the property, for odd function $f(x)$, $\int_{-a}^{a} f(x)dx = 0$,

$\therefore$ we get $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\cos x\,dx = 0$.

Example 9.25

Evaluate : $\int_{-\log 2}^{\log 2} e^{-|x|}\,dx$.

Solution

Let $f(x) = e^{-|x|}$. Then $f(-x) = e^{-|-x|} = e^{-|x|} = f(x)$

So $f(x)$ is an even function.

Hence $$\int_{-\log 2}^{\log 2} e^{-|x|}\,dx = 2\int_{0}^{\log 2} e^{-|x|}\,dx = 2\int_{0}^{\log 2} e^{-x}\,dx = 2(-e^{-x})_{0}^{\log 2} = 2(-e^{-\log 2}+e^{0}) = 2\left(-e^{\log\frac{1}{2}}+1\right)$$

$$= 2\left(-\frac{1}{2}+1\right)=1.$$

Example 9.26

Evaluate : $\displaystyle\int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx.$

Solution

$$\text{Let } I = \int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx \qquad \text{... (1)}$$

Applying the formula $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ in equation (1), we get

$$I = \int_0^a \frac{f(a-x)}{f(a-x)+f(a-(a-x))}\,dx$$

$$= \int_0^a \frac{f(a-x)}{f(x)+f(a-x)}\,dx . \qquad \text{... (2)}$$

Adding equations (1) and (2), we get

$$2I = \int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx + \int_0^a \frac{f(a-x)}{f(x)+f(a-x)}\,dx$$

$$= \int_0^a \frac{f(x)+f(a-x)}{f(x)+f(a-x)}\,dx$$

$$= \int_0^a dx = a .$$

Hence, we get $I = \dfrac{a}{2}$.

Example 9.27

Prove that $\displaystyle\int_0^{\frac{\pi}{4}} \log(1+\tan x)dx = \frac{\pi}{8}\log 2.$

Solution

$$\text{Let us put } I = \int_0^{\frac{\pi}{4}} \log(1+\tan x)dx \qquad \text{... (1)}$$

Applying the property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ in equation (1), we get

$$I = \int_0^{\frac{\pi}{4}} \log\left[1+\tan\left(\frac{\pi}{4}-x\right)\right]dx = \int_0^{\frac{\pi}{4}} \log\left[1+\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}\right]dx$$

$$= \int_0^{\frac{\pi}{4}} \log\left[1+\frac{1-\tan x}{1+\tan x}\right]dx = \int_0^{\frac{\pi}{4}} \log\left[\frac{1+\tan x+1-\tan x}{1+\tan x}\right]dx$$

$$= \int_0^{\frac{\pi}{4}} \log\left[\frac{2}{1+\tan x}\right] dx = \int_0^{\frac{\pi}{4}} [\log 2 - \log(1+\tan x)]\, dx$$

$$= \log 2\int_0^{\frac{\pi}{4}} dx - \int_0^{\frac{\pi}{4}} \log(1+\tan x) dx$$

$$= \frac{\pi}{4}\log 2 - I$$

So, we get $2I = \frac{\pi}{4}\log 2$. Hence, we get $I = \frac{\pi}{8}\log 2$.

Example 9.28

Show that $\int_0^1 \left(\tan^{-1} x + \tan^{-1}(1-x)\right) dx = \frac{\pi}{2} - \log_e 2$.

Solution

$$I = \int_0^1 \left(\tan^{-1} x + \tan^{-1}(1-x)\right) dx$$

$$= \int_0^1 \tan^{-1} x\, dx + \int_0^1 \tan^{-1}(1-x)\, dx$$

$$= \int_0^1 \tan^{-1} x\, dx + \int_0^1 \tan^{-1}(1-(1-x))\, dx, \text{ since } \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

$$= \int_0^1 \tan^{-1} x\, dx + \int_0^1 \tan^{-1} x\, dx$$

$$= 2\int_0^1 \tan^{-1} x\, dx$$

$$= \left[2\int u dv\right]_0^1, \text{ where } u = \tan^{-1} x \text{ and } dv = dx$$

$$= 2\left[uv - \int v du\right]_0^1, \text{ applying integration by parts}$$

$$= 2\left(x\tan^{-1} x - \int x\frac{dx}{1+x^2}\right)_0^1 = 2\left(x\tan^{-1} x - \frac{1}{2}\log\left(1+x^2\right)\right)_0^1 = \frac{\pi}{2} - \log 2$$

Example 9.29

Evaluate $\int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}\, dx$.

Solution

Let us put $I = \int_2^3 \frac{\sqrt{x}}{\sqrt{5-x}+\sqrt{x}}\, dx$ ... (1)

Applying the formula $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$, we get

$$I = \int_2^3 \frac{\sqrt{(2+3-x)}}{\sqrt{5-(2+3-x)}+\sqrt{(2+3-x)}}\, dx = \int_2^3 \frac{\sqrt{5-x}}{\sqrt{x}+\sqrt{5-x}}\, dx \quad \text{... (2)}$$

Adding (1) and (2), we get

$$2I = \int_2^3 \frac{\sqrt{x}+\sqrt{5-x}}{\sqrt{x}+\sqrt{5-x}}\,dx = \int_2^3 dx = [x]_2^3 = 3-2 = 1.$$

Hence, we get $I = \frac{1}{2}$.

Example 9.30

Evaluate $\int\limits_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x}\,dx$

Solution

$$\text{Let } I = \int\limits_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x}\,dx \quad \ldots (1)$$

Using $\int\limits_a^b f(x)\,dx = \int\limits_a^b f(a+b-x)\,dx$ we get,

$$I = \int\limits_{-\pi}^{\pi} \frac{\cos^2(\pi-\pi-x)}{1+a^{\pi-\pi-x}}\,dx$$

$$= \int\limits_{-\pi}^{\pi} \frac{\cos^2(-x)}{1+a^{-x}}\,dx$$

$$= \int\limits_{-\pi}^{\pi} a^x\left(\frac{\cos^2 x}{a^x+1}\right)dx \quad \ldots (2)$$

Adding (1) and (2) we get

$$2I = \int\limits_{-\pi}^{\pi} \frac{\cos^2 x}{a^x+1}(a^x+1)\,dx = \int\limits_{-\pi}^{\pi} \cos^2 x\,dx$$

$$= 2\int\limits_0^{\pi} \cos^2 x\,dx \text{ (since } \cos^2 x \text{ is an even function)}$$

$$\text{Hence } I = \int\limits_0^{\pi} \frac{(1+\cos 2x)}{2}\,dx = \frac{1}{2}\left[x+\frac{\sin 2x}{2}\right]_0^{\pi} = \frac{1}{2}[\pi] = \frac{\pi}{2}.$$

EXERCISE 9.3

1. Evaluate the following definite integrals :

(i) $\int\limits_3^4 \frac{dx}{x^2-4}$ (ii) $\int\limits_{-1}^1 \frac{dx}{x^2+2x+5}$ (iii) $\int\limits_0^1 \sqrt{\frac{1-x}{1+x}}\,dx$

(iv) $\int\limits_0^{\frac{\pi}{2}} e^x\left(\frac{1+\sin x}{1+\cos x}\right)dx$ (v) $\int\limits_0^{\frac{\pi}{2}} \sqrt{\cos\theta}\,\sin^3\theta\,d\theta$ (vi) $\int\limits_0^1 \frac{1-x^2}{\left(1+x^2\right)^2}\,dx$

2. Evaluate the following integrals using properties of integration :

(i) $\int_{-5}^{5} x\cos\left(\frac{e^x-1}{e^x+1}\right)dx$ (ii) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x^5+x\cos x+\tan^3 x+1)\,dx$

(iii) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2 x\ dx$ (iv) $\int_{0}^{2\pi} x\log\left(\frac{3+\cos x}{3-\cos x}\right)dx$

(v) $\int_{0}^{2\pi} \sin^4 x\cos^3 x\,dx$ (vi) $\int_{0}^{1} |5x-3|\ dx$

(vii) $\int_{0}^{\sin^2 x} \sin^{-1}\sqrt{t}\ dt + \int_{0}^{\cos^2 x} \cos^{-1}\sqrt{t}\ dt$ (viii) $\int_{0}^{1} \frac{\log(1+x)}{1+x^2}\ dx$

(ix) $\int_{0}^{\pi} \frac{x\sin x}{1+\sin x}dx$ (x) $\int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{1}{1+\sqrt{\tan x}}dx$

(xi) $\int_{0}^{\pi} x\left[\sin^2(\sin x)+\cos^2(\cos x)\right]dx$

9.4 Bernoulli's Formula

The evaluation of an indefinite integral of the form $\int u(x)v(x)dx$ becomes very simple, when u is a polynomial function of x (that is, $u(x)=a_0x^n+a_1x^{n-1}+\cdots+a_n$) and $v(x)$ can be easily integrated successively. It is accomplished by a formula called **Bernoulli's formula**. This formula is actually an extension of the formula of integration by parts. To derive the formula, we use the following notation:

$$u^{(1)} = \frac{du}{dx},\quad u^{(2)} = \frac{du^{(1)}}{dx},\quad u^{(3)} = \frac{du^{(2)}}{dx},\ \cdots\cdots$$

$$v_{(1)} = \int v dx,\ v_{(2)} = \int v_{(1)}dx,\ v_{(3)} = \int v_{(2)}dx,\cdots\cdots$$

Then, we have

$$dv_{(1)} = vdx,\ dv_{(2)} = v_{(1)}dx,\ dv_{(3)} = v_{(2)}dx,\ \cdots\cdots$$

Now, by integration by parts, we get

$$\begin{aligned}\int uvdx &= \int udv_{(1)} = uv_{(1)} - \int v_{(1)}du = uv_{(1)} - \int v_{(1)}\frac{du}{dx}dx\\ &= uv_{(1)} - \int u^{(1)}dv_{(2)}\\ &= uv_{(1)} - \left(u^{(1)}v_{(2)} - \int v_{(2)}du^{(1)}\right)\\ &= uv_{(1)} - u^{(1)}v_{(2)} + \int v_{(2)}\frac{du^{(1)}}{dx}dx\end{aligned}$$

$$= uv_{(1)} - u^{(1)}v_{(2)} + \int u^{(2)} dv_{(3)}$$

$$= uv_{(1)} - u^{(1)}v_{(2)} + \left(u^{(2)}v_{(3)} - \int v_{(3)} du^{(2)}\right)$$

$$= uv_{(1)} - u^{(1)}v_{(2)} + u^{(2)}v_{(3)} - \int v_{(3)} du^{(2)} .$$

Proceeding in this way, we get

$$\int uvdx = uv_{(1)} - u^{(1)}v_{(2)} + u^{(2)}v_{(3)} - u^{(3)}v_{(4)} + \cdots .$$

The above result is called the **Bernoulli's formula for integration of product of two functions.**

Note

Since u is a polynomial function of x, the successive derivative $u^{(m)}$ will be zero for some positive integer m and so all further derivatives will be zero only. Hence the right-hand-side of the above formula contains a finite number of terms only.

Example 9.31

Evaluate $\int_0^{\pi} x^2 \cos nx \, dx$, where n is a positive integer.

Solution

Taking $u = x^2$ and $v = \cos nx$, and applying the Bernoulli's formula, we get

$$I = \int_0^{\pi} x^2 \cos nx \, dx = \left[\left(x^2\right)\left(\frac{\sin nx}{n}\right) - (2x)\left(-\frac{\cos nx}{n^2}\right) + (2)\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2\pi(-1)^n}{n^2}, \text{ since } \cos n\pi = (-1)^n \text{ and } \sin n\pi = 0 .$$

Example 9.32

Evaluate : $\int_0^1 \mathrm{e}^{-2x}(1 + x - 2x^3)\, dx$.

Solution

Taking $u = 1 + x - 2x^3$ and $v = \mathrm{e}^{-2x}$, and applying the Bernoulli's formula, we get

$$I = \int_0^1 \mathrm{e}^{-2x}(1 + x - 2x^3)\, dx$$

$$= \left[(1 + x - 2x^3)\left(\frac{e^{-2x}}{-2}\right) - \left(1 - 6x^2\right)\left(\frac{e^{-2x}}{4}\right) + (-12x)\left(\frac{e^{-2x}}{-8}\right) - (-12)\left(\frac{e^{-2x}}{16}\right)\right]_0^1$$

$$= \left[\frac{e^{-2x}}{16}\left(16x^3 + 24x^2 + 16x\right)\right]_0^1$$

$$= \frac{7}{2e^2} .$$

Example 9.33

Evaluate : $\int_0^{2\pi} x^2 \sin nx\, dx$, where n is a positive integer.

Solution

Taking $u = x^2$ and $v = \sin nx$, and applying the Bernoulli's formula, we get

$$I = \int_0^{2\pi} x^2 \sin nx\, dx = \left[\left(x^2\right)\left(-\frac{\cos nx}{n}\right) - (2x)\left(-\frac{\sin nx}{n^2}\right) + (2)\left(\frac{\cos nx}{n^3}\right) \right]_0^{2\pi}$$

$$= \left[\left(4\pi^2\right)\left(-\frac{1}{n}\right) - 0 + (2)\left(\frac{1}{n^3}\right) \right] - \left[0 - 0 + (2)\left(\frac{1}{n^3}\right) \right], \text{ since } \cos 2n\pi = 1 \text{ and } \sin 2n\pi = 0$$

$$= -\frac{4\pi^2}{n} + \frac{2}{n^3} - \frac{2}{n^3} = -\frac{4\pi^2}{n}.$$

Example 9.34

Evaluate : $\int_{-1}^{1} e^{-\lambda x}(1 - x^2)\, dx$.

Solution

Taking $u = 1 - x^2$ and $v = e^{-\lambda x}$, and applying the Bernoulli's formula, we get

$$I = \int_{-1}^{1} e^{-\lambda x}(1 - x^2)\, dx = \left[(1 - x^2)\left(\frac{e^{-\lambda x}}{-\lambda}\right) - (-2x)\left(\frac{e^{-\lambda x}}{\lambda^2}\right) + (-2)\left(\frac{e^{-\lambda x}}{-\lambda^3}\right) \right]_{-1}^{1}$$

$$= 2\left(\frac{e^{-\lambda}}{\lambda^2}\right) + 2\left(\frac{e^{-\lambda}}{\lambda^3}\right) + 2\left(\frac{e^{\lambda}}{\lambda^2}\right) - 2\left(\frac{e^{\lambda}}{\lambda^3}\right)$$

$$= \frac{2}{\lambda^2}\left(e^{\lambda} + e^{-\lambda}\right) - \frac{2}{\lambda^3}\left(e^{\lambda} - e^{-\lambda}\right).$$

EXERCISE 9.4

Evaluate the following:

1. $\int_0^1 x^3 e^{-2x}\, dx$
2. $\int_0^1 \frac{\sin(3\tan^{-1} x)\tan^{-1} x}{1 + x^2}\, dx$
3. $\int_0^{\frac{1}{\sqrt{2}}} \frac{e^{\sin^{-1} x} \sin^{-1} x}{\sqrt{1 - x^2}}\, dx$
4. $\int_0^{\frac{\pi}{2}} x^2 \cos 2x\, dx$

9.5 Improper Integrals

In defining the Riemann integral $\int_a^b f(x)dx$, the interval $[a,b]$ of integration is finite and $f(x)$ is finite at every point in $[a,b]$. In many physical applications, the following types of integrals arise:

$$\int_a^{\infty} f(x)\, dx,\ \int_{-\infty}^{a} f(x)\, dx,\ \int_{-\infty}^{\infty} f(x)\, dx,$$

where a is a real number and $f(x)$ is a continuous function on the interval of integration. They are defined as the limits of Riemann integrals as follows:

(i) $$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$

(ii) $$\int_{-\infty}^{a} f(x)\,dx = \lim_{t\to-\infty}\int_t^a f(x)\,dx$$

(iii) $$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_{-t}^t f(x)\,dx$$

They are called **improper integrals of first kind. If the limits exist, then the improper integrals are said to be convergent.**

Note

By the Fundamental theorem of integral calculus, there exists a function $F(t)$ such that

$$\int_a^t f(x)\,dx = F(t)-F(a)$$

$$\therefore \int_a^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx = \lim_{t\to\infty}[F(t)-F(a)] = \left[\int f(x)\,dx\right]_a^{\infty}.$$

Example 9.35

Evaluate $\displaystyle\int_b^{\infty}\frac{1}{a^2+x^2}\,dx,\ a>0, b\in\mathbb{R}$.

Solution

We have $\displaystyle\int_b^{\infty}\frac{1}{a^2+x^2}\,dx = \left[\frac{1}{a}\tan^{-1}\frac{x}{a}\right]_b^{\infty} = \frac{1}{a}\tan^{-1}\infty - \frac{1}{a}\tan^{-1}\frac{b}{a} = \frac{1}{a}\left[\frac{\pi}{2}-\tan^{-1}\frac{b}{a}\right]$.

Note

From the above example, we get

(i) $$\int_0^{\infty}\frac{1}{a^2+x^2}\,dx = \frac{1}{a}\left[\frac{\pi}{2}-\tan^{-1}0\right] = \frac{\pi}{2a}.$$

(ii) $$\int_a^{\infty}\frac{1}{a^2+x^2}\,dx = \frac{1}{a}\left[\frac{\pi}{2}-\tan^{-1}1\right] = \frac{1}{a}\left[\frac{\pi}{2}-\frac{\pi}{4}\right] = \frac{\pi}{4a}.$$

(iii) $$\int_{-\infty}^{\infty}\frac{1}{a^2+x^2}\,dx = \lim_{t\to\infty}\int_{-t}^{t}\frac{1}{a^2+x^2}\,dx = \lim_{t\to\infty}2\int_0^t\frac{1}{a^2+x^2}\,dx,$$ since $\frac{1}{a^2+x^2}$ is even function

$$= 2\int_0^{\infty}\frac{1}{a^2+x^2}\,dx = 2\left(\frac{\pi}{2a}\right) = \frac{\pi}{a}.$$

Example 9.36

Evaluate $\displaystyle\int_0^{\frac{\pi}{2}}\frac{dx}{4\sin^2 x+5\cos^2 x}$.

Solution

$$\text{Let } I = \int_0^{\frac{\pi}{2}}\frac{dx}{4x\sin^2 x+5\cos^2 x}$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{4\tan^2 x + 5} dx \quad \left\{ \begin{array}{l}\text{(Dividing both numerator and} \\ \text{denominator by } \cos^2 x).\end{array}\right.$$

Let $u = \tan x$. Then $du = \sec^2 x\, dx$

When $x = 0, u = \tan 0 = 0$

When $x = \frac{\pi}{2}, u = \tan\frac{\pi}{2} = \infty$.

$$\therefore I = \int_0^{\infty} \frac{du}{4u^2+5} \text{ (This is an improper integral)}$$

$$= \frac{1}{4}\int_0^{\infty} \frac{du}{\left[u^2 + \left(\frac{\sqrt{5}}{2}\right)^2\right]} = \frac{1}{4} \times \frac{2}{\sqrt{5}}\left[\tan^{-1}\left(\frac{u}{\frac{\sqrt{5}}{2}}\right)\right]_0^{\infty} = \frac{1}{2\sqrt{5}}(\tan^{-1}\infty - \tan^{-1} 0) = \frac{1}{2\sqrt{5}}\left(\frac{\pi}{2}\right) = \frac{\pi}{4\sqrt{5}}.$$

EXERCISE 9.5

1. Evaluate the following:

(i) $\int_0^{\frac{\pi}{2}} \frac{dx}{1+5\cos^2 x}$ (ii) $\int_0^{\frac{\pi}{2}} \frac{dx}{5+4\sin^2 x}$

9.6 Reduction Formulae

Certain definite integrals can be evaluated by an index-reduction method. In this section, we obtain the values of the following definite integrals:

$\int_0^{\frac{\pi}{2}} \sin^n x\, dx$, $\int_0^{\frac{\pi}{2}} \cos^n x\, dx$, $\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x\, dx$, $\int_0^{1} x^m (1-x)^n dx$.

We also obtain the value of the improper integral $\int_0^{\infty} e^{-x} x^n\, dx$.

The method of obtaining a reduction formula has the following steps:

Step 1 : Identify an index (positive integer) n in the integral.

Step 2 : Put the integral as I_n.

Step 3 : Applying integration by parts, obtain the equation for I_n in terms of I_{n-1} or I_{n-2}.

The resulting equation is called the reduction formula for I_n.

We list below a few reduction formulae without proof:

Reduction Formula I : If $I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx$, then $I_n = \frac{(n-1)}{n} I_{n-2}, n \geq 2$.

Reduction Formula II : If $I_n = \int_0^{\frac{\pi}{2}} \cos^n x dx$, then $I_n = \frac{(n-1)}{n} I_{n-2}$ $n \geq 2$.

Reduction Formula III : If $I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^m x \cos^n x\, dx$, then $I_{m,n} = \frac{(n-1)}{m+n} I_{m,n-2}, n \geq 2$.

Reduction Formula IV : If $I_{m,n} = \int_0^{1} x^m\, (1-x)^n dx$, then $I_{m,n} = \frac{n}{m+n+1} I_{m,n-1}, n \geq 1$.

Using the reduction formulas I and II, we obtain the following result (stated without proofs):

$$\int_0^{\frac{\pi}{2}} \sin^n x\, dx = \int_0^{\frac{\pi}{2}} \cos^n x\, dx = \begin{cases} \dfrac{(n-1)}{n} \times \dfrac{(n-3)}{(n-2)} \times \cdots \times \dfrac{1}{2} \times \dfrac{\pi}{2}, & \text{if } n = 2,4,6,\cdots \\ \dfrac{(n-1)}{n} \times \dfrac{(n-3)}{(n-2)} \times \cdots \times \dfrac{2}{3}, & \text{if } n = 3,5,7,\cdots \end{cases}$$

Note

As illustrations, we have

$$\int_0^{\frac{\pi}{2}} \cos^5 x\, dx = \int_0^{\frac{\pi}{2}} \sin^5 x\, dx = \frac{4}{5} \times \frac{2}{3} \times 1$$

$$\int_0^{\frac{\pi}{2}} \sin^6 x\, dx = \int_0^{\frac{\pi}{2}} \cos^6 x\, dx = \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$$

Example 9.37

Evaluate $\int_0^{\frac{\pi}{2}} \left(\sin^2 x + \cos^4 x\right) dx$

Solution

Given that $I = \int_0^{\frac{\pi}{2}} \left(\sin^2 x + \cos^4 x\right) dx = \int_0^{\frac{\pi}{2}} \sin^2 x\, dx + \int_0^{\frac{\pi}{2}} \cos^4 x\, dx = \frac{1}{2} \times \frac{\pi}{2} + \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} = \frac{7\pi}{16}$.

Example 9.38

Evaluate $\int_0^{\frac{\pi}{2}} \begin{vmatrix} \cos^4 x & 7 \\ \sin^5 x & 3 \end{vmatrix} dx$.

Solution

$$I = \int_0^{\frac{\pi}{2}} \left(3\cos^4 x - 7\sin^5 x\right) dx = 3\int_0^{\frac{\pi}{2}} \cos^4 x\, dx - 7\int_0^{\frac{\pi}{2}} \sin^5 x\, dx$$

$$= 3 \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} - 7 \times \frac{4}{5} \times \frac{2}{3} = \frac{9\pi}{16} - \frac{56}{15}.$$

By applying the reduction formula III iteratively, we get the following results (stated without proof):

(i) If n is even and m is even,

$$\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x\, dx = \frac{(n-1)}{(m+n)} \frac{(n-3)}{(m+n-2)} \frac{(n-5)}{(m+n-4)} \cdots \frac{1}{(m+2)} \frac{(m-1)}{m} \frac{(m-3)}{(m-2)} \frac{(m-5)}{(m-4)} \cdots \frac{1}{2} \frac{\pi}{2}$$

(ii) If n is odd and m is any positive integer (even or odd), then

$$\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x\, dx = \frac{(n-1)}{(m+n)} \frac{(n-3)}{(m+n-2)} \frac{(n-5)}{(m+n-4)} \cdots \frac{2}{(m+3)} \frac{1}{(m+1)}.$$

Note

If one of m and n is odd, then it is convenient to get the power of $\cos x$ as odd. For instance, if m is odd and n is even, then

$$\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x\, dx = \int_0^{\frac{\pi}{2}} \sin^n x \cos^m x\, dx = \frac{(m-1)}{(n+m)} \frac{(m-3)}{(n+m-2)} \frac{(m-5)}{(n+m-4)} \cdots \frac{2}{(n+3)} \frac{1}{(n+1)}.$$

Example 9.39

Find the values of the following:

(i) $\int_0^{\frac{\pi}{2}} \sin^5 x \cos^4 x\, dx$ (ii) $\int_0^{\frac{\pi}{2}} \sin^4 x \cos^6 x\, dx$

Solution

(i) $$\int_0^{\frac{\pi}{2}} \sin^4 x \cos^6 x \, dx = \frac{(6-1)}{(6+4)} \cdot \frac{(6-3)}{(6+4-2)} \cdot \frac{(6-5)}{(6+4-4)} \cdot \frac{(4-1)}{(4)} \cdot \frac{(4-3)}{(4-2)} \cdot \frac{\pi}{2}$$

$$= \frac{(5)}{(10)} \frac{(3)}{(8)} \frac{(1)}{(6)} \frac{(3)}{(4)} \frac{(1)}{(2)} \frac{\pi}{2} = \frac{3\pi}{512}$$

Also, $$\int_0^{\frac{\pi}{2}} \sin^4 x \cos^6 x \, dx = \int_0^{\frac{\pi}{2}} \sin^6 x \cos^4 x \, dx = \frac{(3)}{(10)} \frac{(1)}{(8)} \frac{(5)}{(6)} \frac{(3)}{(4)} \frac{(1)}{(2)} \frac{\pi}{2} = \frac{3\pi}{512}$$

(ii) $$\int_0^{\frac{\pi}{2}} \sin^5 x \cos^4 x \, dx = \frac{(3)}{(9)} \frac{(1)}{(7)} \frac{(4)}{(5)} \frac{(2)}{(3)} = \frac{(4)}{(9)} \frac{(2)}{(7)} \frac{(1)}{(5)} = \frac{8}{315}$$

Also, $$\int_0^{\frac{\pi}{2}} \sin^5 x \cos^4 x \, dx = \int_0^{\frac{\pi}{2}} \sin^4 x \cos^5 x \, dx = \frac{(4)}{(9)} \frac{(2)}{(7)} \frac{(1)}{(5)} = \frac{8}{315}$$

Example 9.40

Evaluate $\int_0^{2a} x^2 \sqrt{2ax - x^2}\, dx$.

Solution

Put $x = 2a\cos^2\theta$. Then, $dx = -4a\cos\theta\sin\theta d\theta$.

When $x = 0$, $2a\cos^2\theta = 0$ and so $\theta = \frac{\pi}{2}$. When $x = 2a$, $2a\cos^2\theta = 2a$ and so $\theta = 0$.

Hence, we get

$$I = \int_0^{2a} x^2 \sqrt{2ax - x^2}\, dx$$

$$= \int_{\frac{\pi}{2}}^{0} 4a^2\cos^2\theta\sqrt{4a^2\cos^2\theta - 4a^2\cos^4\theta}(-4a\cos\theta\sin\theta)d\theta$$

$$= \int_0^{\frac{\pi}{2}} 4a^2\cos^2\theta 2a\cos\theta\sin\theta(4a\cos\theta\sin\theta)d\theta$$

$$= 32a^4\int_0^{\frac{\pi}{2}} \cos^4\theta\sin^2\theta\, d\theta$$

$$= 32a^4 \times \frac{1}{6} \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} = \pi a^4.$$

Example 9.41

Evaluate $\int_0^1 x^5 \left(1 - x^2\right)^5 dx$.

Solution

Put $x = \sin\theta$. Then, $dx = \cos\theta d\theta$.

When $x = 0$, $\sin\theta = 0$ and so $\theta = 0$. When $x = 1$, $\sin\theta = 1$ and so $\theta = \frac{\pi}{2}$.

Hence, we get

$$I = \int_0^{\frac{\pi}{2}} \sin^5\theta\left(1 - \sin^2\theta\right)^5 \cos\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^5\theta\cos^{11}\theta\, d\theta = \frac{10}{16} \times \frac{8}{14} \times \frac{6}{12} \times \frac{4}{10} \times \frac{2}{8} \times \frac{1}{6} = \frac{1}{336}.$$

By applying the reduction formula III iteratively, we get the following results (stated without proof):

$$\int_0^1 x^m\,(1-x)^n\,dx = \frac{m!\times n!}{(m+n+1)!},\ \text{where } m \text{ and } n \text{ are positive integers.}$$

Example 9.42

Evaluate $\int_0^1 x^3(1-x)^4 dx$.

Solution

$$\int_0^1 x^m(1-x)^n dx = \frac{m!\times n!}{(m+n+1)!}.$$

$$\therefore \int_0^1 x^3(1-x)^4 dx = \frac{3!\times 4!}{(3+4+1)!} = \frac{3!\times 4!}{8!} = \frac{3\times2\times1\times4\times3\times2\times1}{8\times7\times6\times5\times4\times3\times2\times1} = \frac{1}{280}.$$

EXERCISE 9.6

1. Evaluate the following:

(i) $\int_0^{\frac{\pi}{2}} \sin^{10} x\,dx$ (ii) $\int_0^{\frac{\pi}{2}} \cos^7 x\,dx$ (iii) $\int_0^{\frac{\pi}{4}} \sin^6 2x\,dx$ (iv) $\int_0^{\frac{\pi}{6}} \sin^5 3x\,dx$

(v) $\int_0^{\frac{\pi}{2}} \sin^2 x\cos^4 x\,dx$ (vi) $\int_0^{2\pi} \sin^7 \frac{x}{4}\,dx$ (vii) $\int_0^{\frac{\pi}{2}} \sin^3\theta\cos^5\theta\,d\theta$ (viii) $\int_0^1 x^2(1-x)^3\,dx$

9.7 Gamma Integral

In this section, we study about a special improper integral of the form $\int_0^\infty e^{-x}x^{n-1}dx$, where n is a positive integer. Here, we have

$$e^{\infty} = \lim_{x\to\infty} e^x = \infty \text{ and } e^{-\infty} = \lim_{x\to\infty} e^{-x} = \frac{1}{\lim\limits_{x\to\infty} e^x} = \frac{1}{\infty} = 0$$

By L'Hôpital's rule, for every positive integer m, we get,

$$\lim_{x\to\infty} x^m e^{-x} = \lim_{x\to\infty}\frac{x^m}{e^x} = \lim_{x\to\infty}\frac{m!}{e^x} = 0.$$

Example 9.43

Prove that $\int_0^\infty e^{-x}x^n dx = n!$, where n is a positive integer.

Solution

Applying integration by parts, we get

$$\int_0^\infty e^{-x}x^n dx = \left[x^n\left(-e^{-x}\right)\right]_0^\infty - \int_0^\infty \left(-e^{-x}\right)\left(nx^{n-1}\right)dx = n\int_0^\infty e^{-x}x^{n-1}dx.$$

Let $I_n = \int_0^\infty e^{-x}x^n dx$. Then, $I_n = nI_{n-1}$.

So, we get $I_n = n(n-1)I_{n-2}$.

Proceeding in this way, we get ultimately,

$$I_n = n(n-1)(n-2)\cdots(2)(1)I_0 .$$

But, $I_0 = \int_0^{\infty} e^{-x}x^0 dx = \left(-e^{-x}\right)_0^{\infty} = 0+1=1$. So, we get $I_n = n(n-1)(n-2)\cdots(2)(1) = n!$.

Hence, we get

Result

$\int_0^{\infty} e^{-x}x^n dx = n!$, where n is a nonnegative integer.

Note

The integral $\int_0^{\infty} e^{-x}x^{n-1}dx$ defines a unique positive integer for every positive integer $n \geq 1$.

Definition 9.1

$\int_0^{\infty} e^{-x}x^{n-1}dx$ is called the **gamma integral**. It is denoted by $\Gamma(n)$ and is read as "gamma of n".

Note

$$\begin{aligned}
\Gamma(n+1) &= n\Gamma(n) . \\
\Gamma(1) &= \int_0^{\infty} e^{-x}x^0 dx = \left(-e^{-x}\right)_0^{\infty} = 0+1=1 , \\
\Gamma(n) &= \int_0^{\infty} e^{-x}x^{n-1}dx . \\
&= (n-1)!, \; n = 1, 2, 3, \ldots
\end{aligned}$$

Example 9.44

Evaluate $\int_0^{\infty} e^{-ax}x^n dx$, where $a > 0$.

Solution

Making the substitution $t = ax$, we get $dt = adx$ and $x = 0 \Rightarrow t = 0$ and $x = \infty \Rightarrow t = \infty$.

Hence, we get

$$\begin{aligned}
\int_0^{\infty} e^{-ax}x^n dx &= \int_0^{\infty} e^{-t}\left(\frac{t}{a}\right)^n \frac{dt}{a} = \frac{1}{a^{n+1}}\int_0^{\infty} e^{-t}t^n dt \\
&= \frac{1}{a^{n+1}}\int_0^{\infty} e^{-x}x^n dx = \frac{n!}{a^{n+1}} .
\end{aligned}$$

Thus

$$\int_0^{\infty} e^{-ax}x^n dx = \frac{n!}{a^{n+1}}$$

Example 9.45

Show that $\Gamma(n) = 2\int_0^{\infty} e^{-x^2}x^{2n-1}dx.$

Solution

Using the substitution $x = \sqrt{u}$, we get $dx = \dfrac{1}{2\sqrt{u}}du$.

When $x = 0$, we get $u = 0$. When $x = \infty$, we get $u = \infty$.

$$\therefore 2\int_0^{\infty} e^{-x^2}x^{2n-1}dx = 2\int_0^{\infty} e^{-u}\left(\sqrt{u}\right)^{2n-1}\frac{1}{2\sqrt{u}}du = \int_0^{\infty} e^{-u}u^{n-1}du = \Gamma(n) .$$

Example 9.46

Evaluate $\int_0^{\infty} \frac{x^n}{n^x} dx$, where n is a positive integer ≥ 2.

Solution

Using the formula $n = e^{\log_e n}$, we get

$$I = \int_0^{\infty} \frac{x^n}{n^x} dx = \int_0^{\infty} n^{-x} x^n dx = \int_0^{\infty} \left(e^{\log n}\right)^{-x} x^n dx = \int_0^{\infty} e^{-x \log n} x^n dx.$$

Using the substitution $u = x \log n$, we get $dx = \frac{du}{\log n}$.

When $x = 0$, we get $u = 0$. When $x = \infty$, we get $u = \infty$.

$$\therefore I = \int_0^{\infty} e^{-u} \left(\frac{u}{\log n}\right)^n \frac{du}{\log n}$$

$$= \frac{1}{(\log n)^{n+1}} \int_0^{\infty} e^{-u} u^{(n+1)-1} du = \frac{\Gamma(n+1)}{(\log n)^{n+1}} = \frac{n!}{(\log n)^{n+1}}.$$

EXERCISE 9.7

I2K5G4

Evaluate the following

1. (i) $\int_0^{\infty} x^5 e^{-3x} dx$ (ii) $\int_0^{\frac{\pi}{2}} \frac{e^{-\tan x}}{\cos^6 x} dx$

2. If $\int_0^{\infty} e^{-\alpha x^2} x^3 dx = 32$, $\alpha > 0$, find α

9.8 Evaluation of a Bounded Plane Area by Integration

In the beginning of this chapter, we have already introduced definite integral by a geometrical approach. In that approach, we have noted that, whenever the integrand of the definite integral is non-negative, the definite integral yields the geometrical area. In the present section, we apply the approach for finding areas of plane regions bounded by plane curves.

9.8.1 Area of the region bounded by a curve, x – axis and the lines $x = a$ and $x = b$.

Case (i)

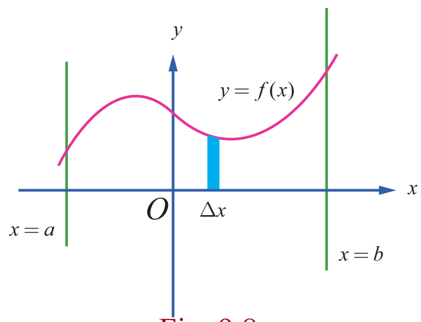


Fig. 9.8

Let $y = f(x)$, $a \leq x \leq b$ be the equation of the portion of the continuous curve that lies above the $x-$axis (that is, the portion lies either in the first quadrant or in the second quadrant) between the lines $x = a$ and $x = b$. See Fig.9.8. Then, $y \geq 0$ for every point of the portion of the curve. Consider the region bounded by the curve, $x-$axis, the ordinates $x = a$ and $x = b$. It is important to note that y does not change its sign in the region. Then, the area A of the region is found as follows:

Viewing in the positive direction of the $y-$axis, divide the region into elementary vertical strips (thin rectangles) of height y and width Δx. Then, A is the limit sum of the areas of the vertical strips. Hence, we get $A = \lim \sum\limits_{a \le x \le b} -y\Delta x = -\int_a^b y\,dx = \left|\int_a^b y\,dx\right|$.

Case (ii)

Let $y = f(x),\ a \le x \le b$ be the equation of the portion of the continuous curve that lies below the $x-$axis (that is, the portion lies either in the third quadrant or in the fourth quadrant). Then, $y \le 0$ for every point of the portion of the curve. It is important to note that y does not change its sign in the region. Consider the region bounded by the curve, $x-$axis, the ordinates $x = a$ and $x = b$. See Fig.9.9. Then, the area A of the region is found as follows:

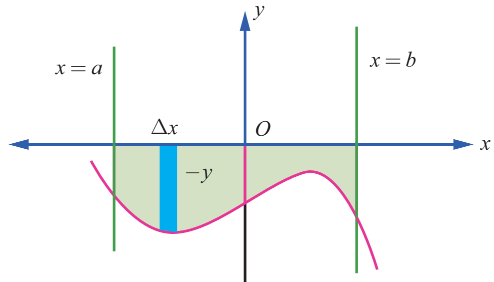


Fig. 9.9

Viewing in the negative direction of the $y-$axis, divide the region into elementary vertical strips (thin rectangles) of height $|y| = -y$ and width Δx. Then, A is the limit of the sum of the areas of the vertical strips. Hence, we get $A = \lim \sum\limits_{a \le x \le b} -y\Delta x = -\int_a^b y\,dx = \left|\int_a^b y\,dx\right|$.

Case (iii)

Let $y = f(x),\ a \le x \le b$ be the equation of the portion of the continuous curve that lies above as well as below the $x-$axis (that is, the portion may lie in all quadrants). Draw the graph of $y = f(x)$ in the $XY-$ plane. The graph lies alternately above and below the $x-$axis and it is intercepted between the ordinates $x = a$ and $x = b$. Divide the interval $[a, b]$ into subintervals $[a, c_1]$, $[c_1, c_2]$, $\cdots$, $[c_k, b]$ such that $f(x)$ has the same sign on each of subintervals. Applying cases (i) and (ii), we can obtain individually, the geometrical areas of the regions corresponding to the subintervals.

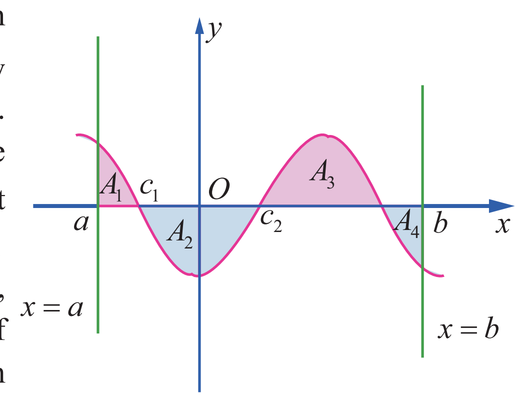


Fig. 9.10

Hence the geometrical area of the region bounded by the graph of $y = f(x)$, the x-axis, the lines $x = a$ and $x = b$ is given by $\left|\int\limits_a^{c_1} f(x)dx\right| + \left|\int\limits_{c_1}^{c_2} f(x)dx\right| + \cdots + \left|\int\limits_{c_k}^{b} f(x)dx\right|$.

For instance, consider the shaded region in Fig. 9.10. Here A_1, A_2, A_3, and A_4 denote geometric areas of the individual parts. Then, the total area is given by

$$A = A_1 + A_2 + A_3 + A_4 = \int\limits_a^{c_1} f(x)dx + \left|\int\limits_{c_1}^{c_2} f(x)dx\right| + \int\limits_{c_2}^{c_3} f(x)dx + \left|\int\limits_{c_3}^{b} f(x)dx\right|.$$

9.8.2 Area of the region bounded by a curve, y– axis and the lines *y* = *c* and *y* = *d*.

Case (iv)

Let $x = f(y),\ c \le y \le d$ be the equation of the portion of the continuous curve that lies to the right side of $y-$axis (that is, the portion lies either in the first quadrant or in the fourth quadrant). Then, $x \ge 0$ for every point of the portion of the curve. It is important to note that x does not change its sign in the region.

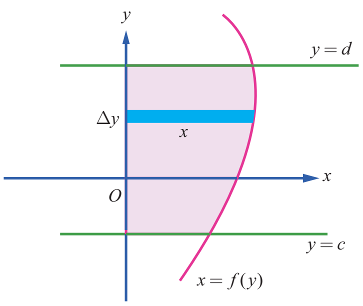


Fig. 9.11

Consider the region bounded by the curve, $y-$axis, the lines $y = c$ and $y = d$. The region is sketched as in Fig. 9.11. Then, the area A of the region is found as follows:

Viewing in the positive direction of the $x-$axis, divide the region into thin horizontal strips (thin rectangles) of length x and width Δy. Then, A is the limit of the sum of the areas of the horizontal strips. Hence, we get $A = \lim \sum\limits_{c \le y \le d} x\Delta y = \int_c^d x dy$.

Case (v)

Let $x = f(y),\ c \le y \le d$ be the equation of the portion of the continuous curve that lies to the left side of $y-$axis (that is, the portion lies either in the second quadrant or in the third quadrant). Then, $x \le 0$ for every point of the portion of the curve. It is important to note that x does not change its sign in the region. Consider the region bounded by the curve, $y-$axis, the lines $y = c$ and $y = d$. The region is sketched as in Fig. 9.12. Then, the area A of the region is found as follows:

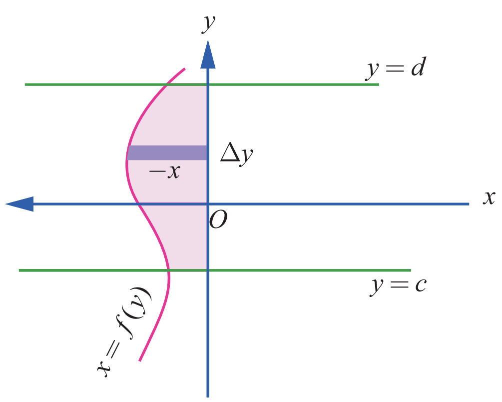


Fig. 9.12

Viewing in the positive direction of the $x-$axis, divide the region into thin horizontal strips (thin rectangles) of length $|x| = -x$ and width Δy. Then, A is the limit of the sum of the areas of the horizontal strips.

Hence, we get $A = \lim \sum\limits_{c \le y \le d} (-x)\Delta y = -\int_c^d x\,dy = \left|\int_c^d x\,dy\right|$.

Case (vi)

Let $x = f(y),\ c \le y \le d$ be the equation of the portion of the continuous curve that lies to the right as well as to the left of the $y-$axis (that is, the portion may lie in all quadrants). Draw the graph of $x = f(y)$ in the $XY-$ plane. The graph lies alternately to the right and to the left of the $y-$axis and it is intercepted between the lines $y = c$ and $y = d$. Divide the interval $[c, d]$ into subintervals $[c, a_1]$, $[a_1, a_2], \cdots$, $[a_k, d]$ such that $f(y)$ has the same sign on each of subintervals. Applying cases

(iii) and (iv), we can obtain individually, the geometrical areas of the regions corresponding to the subintervals.

Hence the geometrical area A of the region bounded by the graph of $x = f(y)$, the y-axis, the lines $y = c$ and $y = d$ is given by
$A = \left|\int_c^{a_1} f(y)dy\right| + \left|\int_{a_1}^{a_2} f(y)dy\right| + \cdots + \left|\int_{a_k}^{d} f(y)dy\right|$.

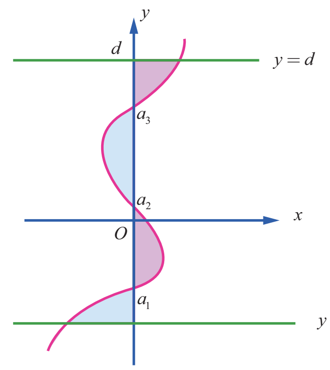


Fig. 9.13

For instance, consider the shaded region in Fig. 9.13. Here, B_1, B_2, B_3 and B_4 denote geometric areas of the individual parts. Then, the total area B of the region bounded by the curve $x = f(y)$, $y-$axis and the lines $y = c$ and $y = d$ is given by

$$B = B_1 + B_2 + B_3 + B_4$$

$$= \left|\int_c^{a_1} f(y)dy\right| + \int_{a_1}^{a_2} f(y)dy + \left|\int_{a_2}^{a_3} f(y)dy\right| + \int_{a_3}^{d} f(y)dy.$$

Example 9.47

Find the area of the region bounded by the line $6x + 5y = 30$, $x-$axis and the lines $x = -1$ and $x = 3$.

Solution

The region is sketched in Fig. 9.14. It lies above the $x-$axis. Hence, the required area is given by

$$A = \int_{-1}^{3} y\,dx = \int_{-1}^{3}\left(\frac{30-6x}{5}\right)dx = \left(\frac{30x-3x^2}{5}\right)_{-1}^{3}$$

$$= \left(\frac{90-27}{5}\right) - \left(\frac{-30-3}{5}\right) = \frac{96}{5}.$$

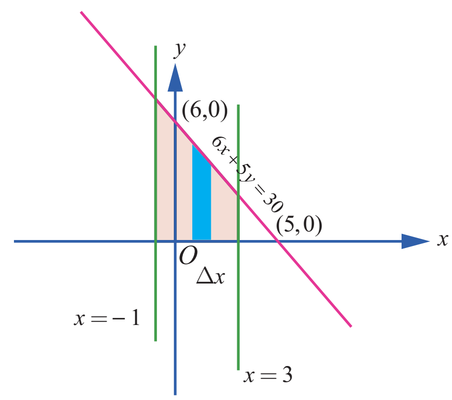


Fig. 9.14

Example 9.48

Find the area of the region bounded by the line $7x - 5y = 35$, $x-$axis and the lines $x = -2$ and $x = 3$.

Solution

The region is sketched in Fig. 9.15. It lies below the $x-$axis. Hence, the required area is given by

$$A = \left|\int_{-2}^{3} y\,dx\right| = \left|\int_{-2}^{3}\left(\frac{7x-35}{5}\right)dx\right|$$

$$= \frac{1}{5}\left|\left(7\left(\frac{x^2}{2}\right) - 35x\right)_{-2}^{3}\right|$$

$$= \frac{1}{5}\left|\left(\left(\frac{63}{2}\right) - 105\right) - (84)\right| = \frac{63}{2}.$$

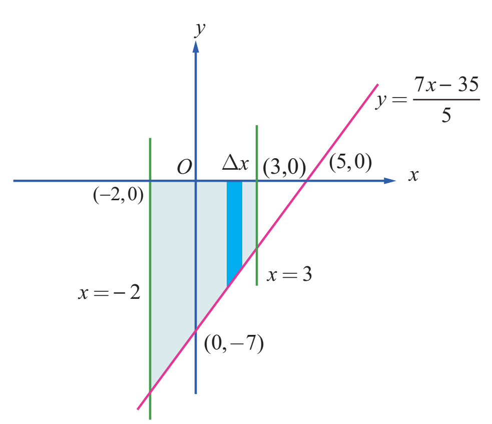


Fig. 9.15

Example 9.49

Find the area of the region bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Solution

The ellipse is symmetric about both major and minor axes. It is sketched as in Fig.9.16. So, viewing in the positive direction of y-axis, the required area A is four times the area of the region bounded by the portion of the ellipse in the first quadrant $\left(y=\frac{b}{a}\sqrt{a^2-x^2}, 0<x<a\right)$, x-axis, $x=0$ and $x=a$.

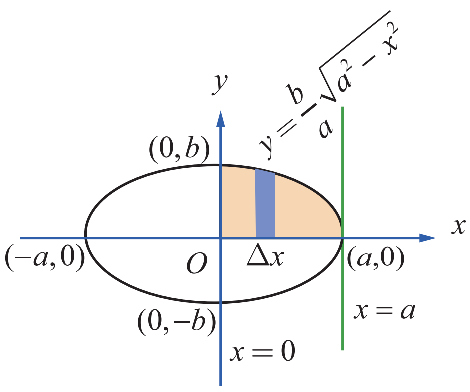


Fig. 9.16

Hence, by taking vertical strips, we get

$$A = 4\int_0^a y\,dx = 4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$$

$$= \frac{4b}{a}\left[\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a = \frac{4b}{a}\times\frac{\pi a^2}{4} = \pi ab$$

Note

Viewing in the positive direction of x-axis, the required area A is four times the area of the region bounded by the portion of the ellipse in the first quadrant $\left(x=\frac{a}{b}\sqrt{b^2-y^2}, 0<y<b\right)$ y-axis, $y=0$ and $y=b$. Hence, by taking horizontal strips (see Fig.9.17), we get

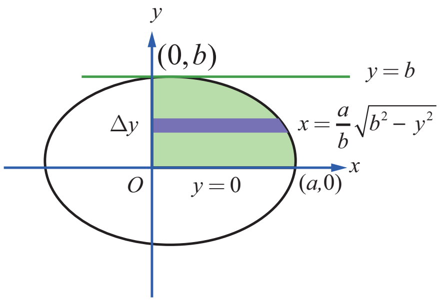


Fig. 9.17

$$A = \int_0^a x\,dy = 4\int_0^b \frac{a}{b}\sqrt{b^2-y^2}\,dy$$

$$= \frac{4a}{b}\left[\frac{y\sqrt{b^2-y^2}}{2}+\frac{b^2}{2}\sin^{-1}\left(\frac{y}{b}\right)\right]_0^b = \frac{4a}{b}\times\frac{\pi b^2}{4} = \pi ab.$$

Note

Putting $b=a$ in the above result, we get that the area of the region enclosed by the circle $x^2+y^2=a^2$ is πa^2.

Example 9.50

Find the area of the region bounded between the parabola $y^2=4ax$ and its latus rectum.

Solution

The equation of the latus-rectum is $x=a$. It intersects the parabola at the points $L(a,2a)$ and $L_1(a,-2a)$. The required area is sketched in Fig. 9.18. By symmetry, the required area A is twice the area bounded by the portion of the parabola

$$y=2\sqrt{a}\sqrt{x}, x\text{-axis}, \; x=0 \;\text{ and }\; x=a.$$

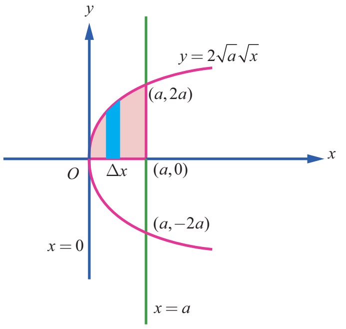


Fig. 9.18

Hence, by taking vertical strips, we get

$$A = 2\int_0^a y\,dx = 2\int_0^a 2\sqrt{a}\sqrt{x}\,dx = 4\sqrt{a}\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^a$$

$$= 4\sqrt{a}\times\frac{2}{3}a^{\frac{3}{2}} = \frac{8a^2}{3}.$$

Note

Viewing in the positive direction of x-axis, and making horizontal strips (see Fig. 9.19), we get

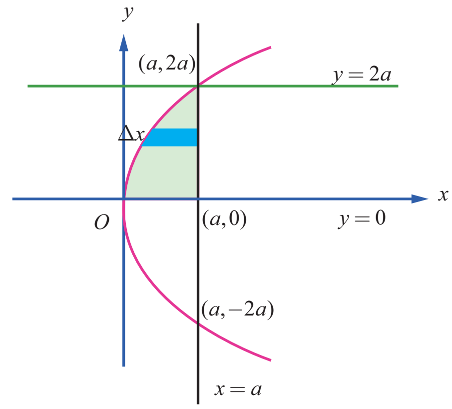


Fig. 9.19

$$A = 2\int_0^{2a}(a-x)dy = 2\int_0^{2a}\left(a-\frac{y^2}{4a}\right)dy$$

$$= 2\left(ay-\frac{y^3}{12a}\right)_0^{2a} = 2\left(2a^2-\frac{8a^3}{12a}\right) = \frac{8a^2}{3}.$$

Note

It is quite interesting to note that the above area is equal to two-thirds the base (latus-rectum) times the height (the distance between the focus and the vertex). This verifies Archimedes' formula for areas of parabolic arches which states that the area under a parabolic arch is two-thirds the area of the rectangle having base of the arch as length and height of the arch as the breadth. It is also equal to four-thirds the area of the triangle with base (latus-rectum) and height (the distance between the focus and the vertex).

Example 9.51

Find the area of the region bounded by the y-axis and the parabola $x = 5-4y-y^2$.

Solution

The equation of the parabola is $(y+2)^2 = -(x-9)$. The parabola crosses the y-axis at $(0,-5)$ and $(0,1)$. The vertex is at $(9,-2)$ and the axis of the parabola is $y=-2$. The required area is sketched as in Fig. 9.20.

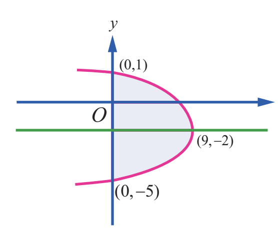


Fig. 9.20

Viewing in the positive direction of $x-$ axis, and making horizontal strips, the required area A is given by

$$A = \int_{-5}^{1} x\,dy = \int_{-5}^{1}(5-4y-y^2)dy = \left[5y-2y^2-\frac{y^3}{3}\right]_{-5}^{1} = \frac{8}{3}-\left(-\frac{100}{3}\right) = 36.$$

Note

As in the previous problem, we again verify Archimedes' formula that the area of the parabolic arch is equal to two-thirds the base times the height.

Example 9.52

Find the area of the region bounded by $x-$ axis, the sine curve $y=\sin x$, the lines $x=0$ and $x=2\pi$.

Solution

The required area is sketched in Fig. 9.21. One portion of the region lies above the $x-$ axis between $x=0$ and $x=\pi$, and the other portion lies below $x-$ axis between $x=\pi$ and $x=2\pi$. So, the required area is given by

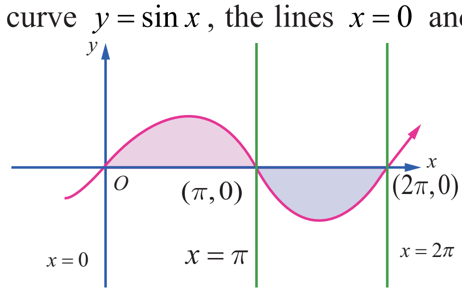


Fig. 9.21

$$A = \int_0^{\pi} y dx + \left|\int_{\pi}^{2\pi} y dx\right| = \int_0^{\pi} \sin x dx + \left|\int_{\pi}^{2\pi} \sin x dx\right| = [-\cos x]_0^{\pi} + \left|[-\cos x]_{\pi}^{2\pi}\right|$$

$$= [-\cos \pi + \cos 0] + \left|[-\cos 2\pi + \cos \pi]\right| = 2 + |-2| = 4.$$

Note

If we compute the definite integral $\int_0^{2\pi} \sin x dx$, we get

$$\int_0^{2\pi} \sin x dx = [-\cos x]_0^{2\pi} = [-\cos 2\pi] - [-\cos 0] = 0.$$

So $\int_0^{2\pi} f(x) dx$ does not represent the area of the region bounded by the curve $y = \sin x$, $x-$axis, the lines $x = 0$ and $x = 2\pi$.

Example 9.53

Find the area of the region bounded by $x-$axis, the curve $y = |\cos x|$, the lines $x = 0$ and $x = \pi$.

Solution

The given curve is $y = \begin{cases} \cos x, 0 \le x \le \dfrac{\pi}{2} \\ -\cos x, \dfrac{\pi}{2} \le x \le \pi \end{cases}$

It lies above the $x-$axis. The required area is sketched in Fig. 9.22. So, the required area is given by

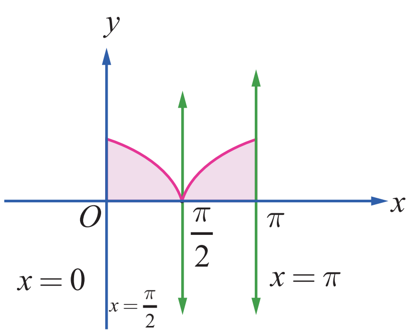


Fig. 9.22

$$A = \int_0^{\pi} y dx = \int_0^{\frac{\pi}{2}} \cos x dx + \int_{\frac{\pi}{2}}^{\pi} (-\cos x) dx = [\sin x]_0^{\frac{\pi}{2}} - [\sin x]_{\frac{\pi}{2}}^{\pi}$$

$$= [1-0] - [0-1] = 2.$$

9.8.3 Area of the region bounded between two curves

Case (i)

Let $y = f(x)$ and $y = g(x)$ be the equations of two curves in the $XOY-$plane such that $f(x) \ge g(x)$ for all $x \in [a,b]$. We want to find the area A of the region bounded between the two curves, the ordinates $x = a$ and $x = b$.

The required area is sketched in Fig. 9.23. To compute A, we divide the region into thin vertical strips of width Δx and height $f(x) - g(x)$. It is important note that $f(x) - g(x) \ge 0$ for all $x \in [a,b]$. As before, the required area is the limit of the sum of the areas of the vertical strips. Hence, we get $A = \int_a^b [f(x) - g(x)] dx$.

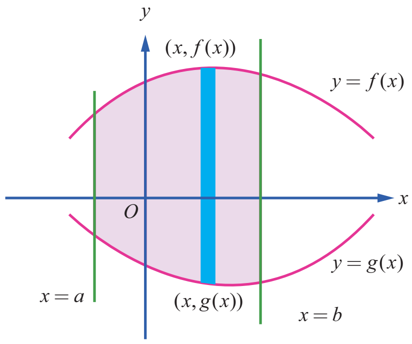


Fig. 9.23

Note

Viewing in the positive direction of $y-$axis, the curve $y = f(x)$ can be termed as the upper curve (U) and the curve $y = g(x)$ as the lower curve (L). Thus, we get $A = \int_a^b [y_U - y_L]dx$.

Case (ii)

Let $x = f(y)$ and $x = g(y)$ be the equations of two curves in the $XOY-$plane such that $f(y) \geq g(y)$ for all $y \in [c,d]$. We want to find the area A of the region bounded between the two curves, the lines $y = c$ and $y = d$. The required area is sketched in Fig. 9.24. To compute A, we view in the positive direction of the $x-$axis and divide the region into thin horizontal strips of width Δy and height $f(y) - g(y)$. It is important note that $f(y) - g(y) \geq 0$ for all $y \in [c,d]$. As before, the required area is the limit of the sum of the areas of the horizontal strips. Hence, we get $A = \int_c^d [f(y) - g(y)]dy$.

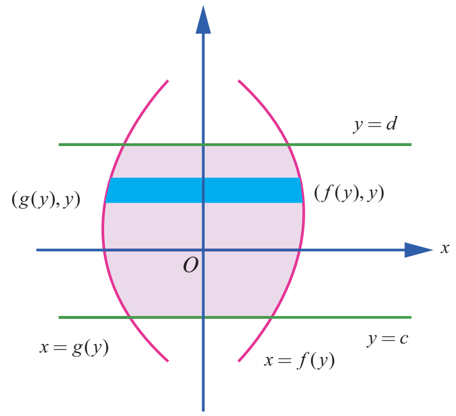


Fig. 9.24

Note

Viewing in the positive direction of $x-$ axis, the curve $x = f(y)$ can be termed as the right curve (R) and the curve $x = f(y)$ as the left curve (L). Thus, we get $A = \int_a^b [x_R - x_L]dy$.

Example 9.54

Find the area of the region bounded between the parabolas $y^2 = 4x$ and $x^2 = 4y$.

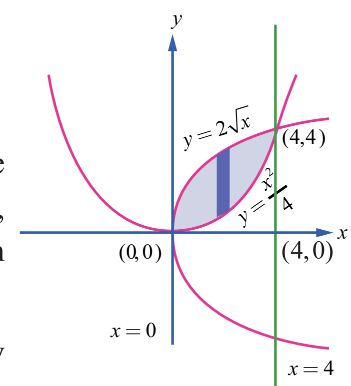


Fig. 9.25

Solution

First, we get the points of intersection of the parabolas. For this, we solve $y^2 = 4x$ and $x^2 = 4y$ simultaneously: Eliminating y between them, we get $x^4 = 64x$ and so $x = 0$ and $x = 4$. Then the points of intersection are $(0,0)$ and $(4,4)$. The required region is sketched in Fig.9.25.

Viewing in the direction of y-axis, the equation of the upper boundary is $y = 2\sqrt{x}$ for $0 \leq x \leq 4$ and the equation of the lower boundary is $y = \dfrac{x^2}{4}$ for $0 \leq x \leq 4$. So, the required area Δ is

$$A = \int_0^4 (y_U - Y_L)\,dx = \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right)dx = \left[2\left(\frac{2x^{3/2}}{3}\right) - \frac{x^3}{12}\right]_0^4 = \left[2\left(\frac{2\times 8}{3}\right) - \frac{64}{12}\right] - 0 = \frac{16}{3}.$$

Note

Viewing in the positive direction of x-axis, the right bounding curve is $x^2 = 4y$ and the left bounding curve is $y^2 = 4x$. See Fig. 9.26. The equation of the right boundary is $x = 2\sqrt{y}$ for $0 \leq y \leq 4$ and the equation of the left boundary is $x = \dfrac{y^2}{4}$ for $0 \leq y \leq 4$. So, the required area A is

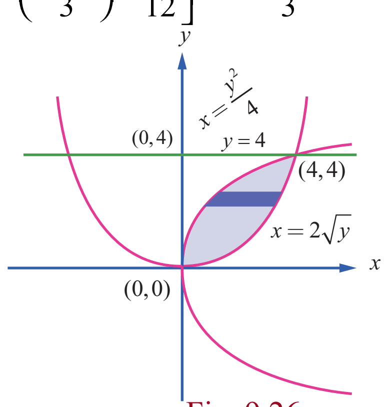


Fig. 9.26

$$A=\int_0^4 (x_R - x_L)\,dx = \int_0^4\left(2\sqrt{y}-\frac{y^2}{4}\right)dy=\left[2\left(\frac{2y^{3/2}}{3}\right)-\frac{y^3}{12}\right]_0^4=\left[2\left(\frac{2\times 8}{3}\right)-\frac{64}{12}\right]-0=\frac{16}{3}\,.$$

Example 9.55

Find the area of the region bounded between the parabola $x^2 = y$ and the curve $y=|x|$.

Solution

Both the curves are symmetrical about y-axis.

The curve $y=|x|$ is $y=\begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$.

It intersects the parabola $x^2 = y$ at $(1,1)$ and $(-1,1)$.

The area of the region bounded by the curves is sketched in Fig. 9.27. It lies in the first quadrant as well as in the second quadrant. By symmetry, the required area is twice the area in the first quadrant.

In the first quadrant, the upper curve is $y = x, 0 \leq x \leq 1$ and the lower curve is $y = x^2, 0 \leq x \leq 1$. Hence, the required area is given by

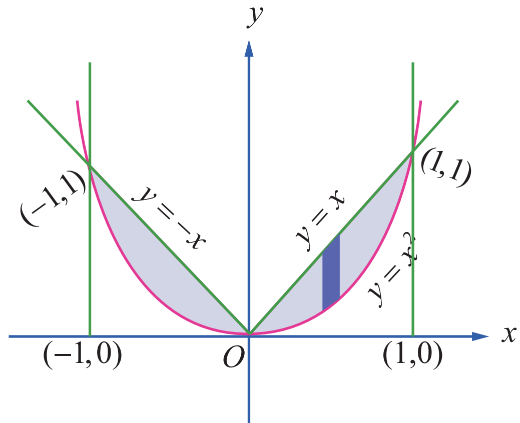


Fig. 9.27

$$A = 2\int_0^1 [y_U - y_L]\,dx = 2\int_0^1 [x - x^2]\,dx$$

$$= 2\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1$$

$$= 2\left(\frac{1}{2}-\frac{1}{3}\right)=\frac{1}{3}.$$

Example 9.56

Find the area of the region bounded by $y=\cos x, y = \sin x$, the lines $x=\frac{\pi}{4}$ and $x=\frac{5\pi}{4}$.

Solution

The region is sketched in Fig. 9.28. The upper boundary of the region is $y=\sin x$ for $\frac{\pi}{4}\leq x \leq \frac{5\pi}{4}$ and the lower boundary of the region is $y = \cos x$ for $\frac{\pi}{4}\leq x\leq\frac{5\pi}{4}$. So the required area A is given by

$$A = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}}\left(y_U - y_L\right)dx=\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}}(\sin x - \cos x)dx = \left[-\cos x - \sin x\right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$

$$=\left(-\sin\frac{5\pi}{4}-\cos\frac{5\pi}{4}\right)-\left(-\sin\frac{\pi}{4}-\cos\frac{\pi}{4}\right)$$

$$=\left(-\left(-\frac{1}{\sqrt{2}}\right)-\left(-\frac{1}{\sqrt{2}}\right)\right)-\left(-\left(\frac{1}{\sqrt{2}}\right)-\left(\frac{1}{\sqrt{2}}\right)\right)$$

$$=\frac{2}{\sqrt{2}}+\frac{2}{\sqrt{2}}=2\sqrt{2}\,.$$

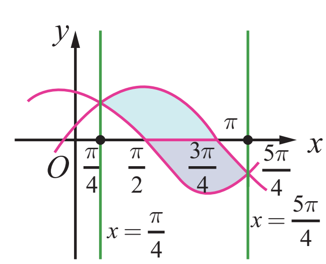


Fig. 9.28

Example 9.57

The region enclosed by the circle $x^2 + y^2 = a^2$ is divided into two segments by the line $x = h$. Find the area of the smaller segment.

Solution

The smaller segment is sketched in Fig. 9.29. Here $0 < h < a$. By symmetry about the x-axis, the area of the smaller segment is given by

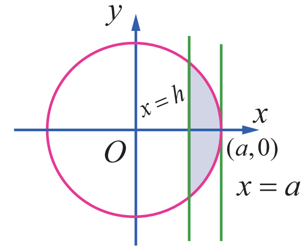


Fig. 9.29

$$A = 2\int_h^a \sqrt{a^2 - x^2}\, dx = 2\left[\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_h^a$$

$$= 2\left[0 + \frac{a^2}{2}\sin^{-1}(1)\right] - 2\left[\frac{h\sqrt{a^2-h^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{h}{a}\right)\right]$$

$$= a^2\left(\frac{\pi}{2}\right) - h\sqrt{a^2-h^2} - a^2\sin^{-1}\left(\frac{h}{a}\right)$$

$$= a^2\left[\frac{\pi}{2} - \sin^{-1}\left(\frac{h}{a}\right)\right] - h\sqrt{a^2-h^2}$$

$$= a^2\cos^{-1}\left(\frac{h}{a}\right) - h\sqrt{a^2-h^2}.$$

■

Example 9.58

Find the area of the region in the first quadrant bounded by the parabola $y^2 = 4x$, the line $x + y = 3$ and y-axis.

Solution

First, we find the points of intersection of $x + y = 3$ and $y^2 - 4x$:

$x + y = 3 \Rightarrow y = 3 - x$.

$\therefore y^2 = 4x \Rightarrow (3-x)^2 = 4x$

$\Rightarrow x^2 - 10x + 9 = 0$

$\Rightarrow x = 1, x = 9$.

$\therefore x = 1$ in $x + y = 3 \Rightarrow y = 2$, and $x = 9$ in $x + y = 3 \Rightarrow y = -6$.

$\therefore (1,2)$ and $(9,-6)$ are the points of intersection.

The line $x + y = 3$ meets the y-axis at $(0,3)$.

The required area is sketched in Fig. 9.30.

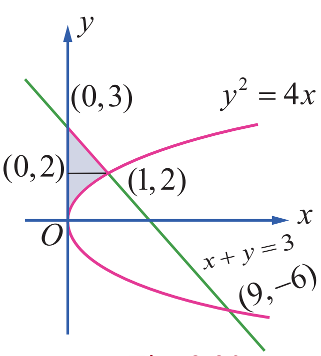


Fig. 9.30

Viewing in the direction of y-axis, on the right bounding curve is given by

$$x = \begin{cases} \dfrac{y^2}{4}, & 0 \le y \le 2 \\ 3 - y, & 2 \le y \le 3 \end{cases}$$

$$\therefore A = \int_0^2 x\,dy + \int_2^3 x\,dy = \int_0^2 \frac{y^2}{4}\,dy + \int_2^3 (3-y)\,dy$$

$$= \left(\frac{y^3}{12}\right)_0^2 + \left(3y - \frac{y^2}{2}\right)_2^3 = \left(\frac{8}{12} - 0\right) + \left(9 - \frac{9}{2}\right) - \left(6 - \frac{4}{2}\right) = \frac{7}{6}.$$

■

Example 9.59

Find, by integration, the area of the region bounded by the lines $5x-2y=15$, $x+y+4=0$ and the x-axis.

Solution

The lines $5x-2y=15$, $x+y+4=0$ intersect at $(1,-5)$. The line $5x-2y=15$ meets the x-axis at $(3,0)$. The line $x+y+4=0$ meets the x-axis at $(-4,0)$. The required area is shaded in Fig.9.31. It lies below the x-axis. It can be computed either by considering vertical strips or horizontal strips.

When we do by vertical strips, the region has to be divided into two sub-regions by the line $x=1$. Then, we get

$$
\begin{aligned}
A &= \left|\int_{-4}^{1} y\,dx\right| + \left|\int_{1}^{3} y\,dx\right| \\
&= \left|\int_{-4}^{1}(-4-x)\,dx\right| + \left|\int_{1}^{3}\left(\frac{5x-15}{2}\right)dx\right| \\
&= \left|\left(-4x-\frac{x^2}{2}\right)_{-4}^{1}\right| + \left|\left(\frac{5x^2}{4}-\frac{15x}{2}\right)_{1}^{3}\right| \\
&= \left|\left(-\frac{9}{2}\right)-(8)\right| + \left|\left(-\frac{45}{4}\right)-\left(-\frac{25}{4}\right)\right| \\
&= \frac{25}{2}+5 \\
&= \frac{35}{2}.
\end{aligned}
$$

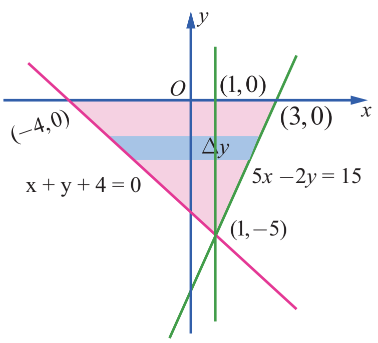


Fig. 9.31

When we do by horizontal strips, there is no need to subdivide the region. In this case, the area is bounded on the right by the line $5x-2y=15$ and on the left by $x+y+4=0$. So, we get

$$
\begin{aligned}
A &= \int_{-5}^{0}[x_R - x_L]\,dy = \int_{-5}^{0}\left[\frac{15+2y}{5}-(-4-y)\right]dy \\
&= \int_{-5}^{0}\left[7+\frac{7y}{5}\right]dy = \left[7y+\frac{7y^2}{10}\right]_{-5}^{0} \\
&= 0-\left[-35+\frac{35}{2}\right] = \frac{35}{2}.
\end{aligned}
$$

Note

The region is triangular with base 7 units and height 5 units. Hence its area is $\frac{35}{2}$ without using integration.

Example 9.60

Using integration find the area of the region bounded by triangle ABC, whose vertices A, B, and C are $(-1,1)$, $(3,2)$, and $(0,5)$ respectively.

Solution

See Fig. 9.32.

Equation of AB is $\dfrac{y-1}{2-1}=\dfrac{x+1}{3+1}$ or $y=\dfrac{1}{4}(x+5)$

Equation of BC is $\dfrac{y-5}{2-5}=\dfrac{x-0}{3-0}$ or $y=-x+5$

Equation of AC is $\dfrac{y-1}{5-1}=\dfrac{x+1}{0+1}$ or $y=4x+5$

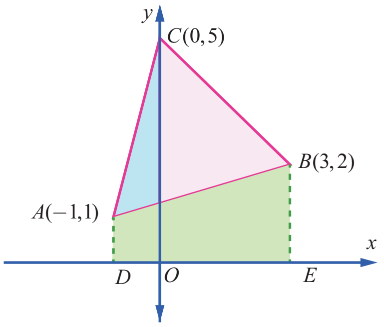


Fig. 9.32

$\therefore$ Area of ΔABC = Area $DACO$+ Area of $OCBE$ − Area of $DABE$

$$=\int_{-1}^{0}(4x+5)\,dx+\int_{0}^{3}(-x+5)\,dx-\frac{1}{4}\int_{-1}^{3}(x+5)\,dx$$

$$=\left[\frac{4x^2}{2}+5x\right]_{-1}^{0}+\left[-\frac{x^2}{2}+5x\right]_{0}^{3}-\frac{1}{4}\left[\frac{x^2}{2}+5x\right]_{-1}^{3}$$

$$=0-(+2-5)+\left(-\frac{9}{2}+15\right)-0-\frac{1}{4}\left[\frac{9}{2}+15\right]+\frac{1}{4}\left[\frac{1}{2}-5\right]=\frac{15}{2}$$

Example 9.61

Using integration, find the area of the region which is bounded by x-axis, the tangent and normal to the circle $x^2+y^2=4$ drawn at $\left(1,\sqrt{3}\right)$.

Solution

We recall that the equation of the tangent to the circle $x^2+y^2=a^2$ at $\left(x_1,y_1\right)$ is $xx_1+yy_1=a^2$. So, the equation of the tangent to the circle $x^2+y^2=4$ at $\left(1,\sqrt{3}\right)$ is $x+y\sqrt{3}=4$; that is, $y=-\dfrac{1}{\sqrt{3}}(x-4)$. The tangent meets the x-axis at the point (4,0).

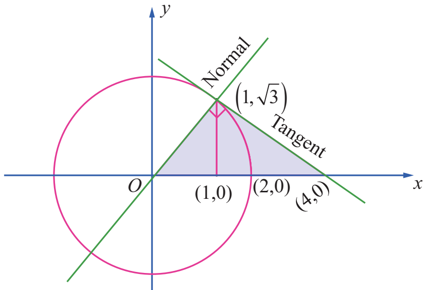


Fig. 9.33

The slope of the tangent is $-\frac{1}{\sqrt{3}}$. So the slope of the normal is $\sqrt{3}$ and hence equation of the normal is $y-\sqrt{3}=\sqrt{3}(x-1)$; that is $y=\sqrt{3}x$ and it passes through the origin. The area to be found is shaded in the adjoining figure. It can be found by two methods.

Method 1

Viewing in the postive direction of y-axis, the required area is the area of the region bounded by x-axis, $y=\sqrt{3}x$ and $x+y\sqrt{3}=4$. So it can be obtained by applying the formula $\int_a^b y\,dx$. For this, we have to split the region into sub-regions, one sub-region bounded by x-axis, the normal $y=\sqrt{3}x$ and the line $x=1$; the other sub-region bounded by x-axis, the tangent $x+y\sqrt{3}=4$ and the line $x=1$ axis.

$$\therefore \text{ Area required } =\int_0^1 y\,dx+\int_1^4 y\,dx=\int_0^1\sqrt{3}x\,dx+\int_1^4\left[-\frac{1}{\sqrt{3}}(x-4)\right]dx$$

$$=\left[\sqrt{3}\frac{x^2}{2}\right]_0^1+\left[-\frac{1}{\sqrt{3}}\left(\frac{x^2}{2}-4x\right)\right]_1^4=\frac{\sqrt{3}}{2}+\frac{8}{\sqrt{3}}-\frac{7}{2\sqrt{3}}=2\sqrt{3}.$$

Method 2

Viewing in the direction of x-axis, the required area is the area of the region bounded between $y=\sqrt{3}x$ and $x+y\sqrt{3}=4$, $y=0$ and $y=\sqrt{3}$. So, it can be obtained by applying the formula

$$\int_c^d (x_R - x_L)\,dy$$

Here, $c=0$, $d=\sqrt{3}$, x_R is the x-value on the tangent $x+y\sqrt{3}=4$ and x_L is the x-value on the normal $y=\sqrt{3}x$.

$$\therefore \text{ Area required } = \int_c^d (x_R - x_L)\,dy = \int_0^{\sqrt{3}} \left(\left(4 - y\sqrt{3}\right) - \frac{y}{\sqrt{3}} \right) dy$$

$$= \left[\left(4y - \frac{y^2}{2}\sqrt{3} \right) - \frac{y^2}{2\sqrt{3}} \right]_0^{\sqrt{3}}$$

$$= 4\sqrt{3} - \frac{3}{2}\sqrt{3} - \frac{3}{2\sqrt{3}} = 2\sqrt{3}.$$

Working rule for finding area of the region bounded by $y = f_1(x)$, $y = f_2(x)$, the lines $x = a$ and $x = b$, where $a < b$:

Draw an arbitrary line parallel to y-axis cutting the plane region. First, find the y-coordinate of the point where the line enters the region. Call it y_{ENTRY}. Next, find the y-coordinate of the point where the line exits the region. Call it y_{EXIT}. Both y_{ENTRY} and y_{EXIT} can be found from the equations of the bounding curves. Then, the required area is given by $\int_a^b [y_{EXIT} - y_{ENTRY}]\,dx$.

Working rule for finding area of the region bounded by $x = g_1(y)$, $x = g_2(y)$, the lines $y = c$ and $y = d$, where $c < d$:

Draw an arbitrary line parallel to x-axis cutting the plane region.

First, find the x-coordinate of the point where the line enters the region. Call it x_{ENTRY} .

Next, find the x-coordinate of the point where the line exits the region. Call it x_{EXIT}. Both x_{ENTRY} and x_{EXIT} can be found from the equations of the bounding curves. Then, the required area is given by $\int_c^d [x_{EXIT} - x_{ENTRY}]\,dy$.

EXERCISE 9.8

1. Find the area of the region bounded by $3x-2y+6=0$, $x=-3$, $x=1$ and x-axis.
2. Find the area of the region bounded by $2x-y+1=0$, $y=-1$, $y=3$ and y-axis.
3. Find the area of the region bounded by the curve $2+x-x^2+y=0$, x-axis, $x=-3$ and $x=3$.
4. Find the area of the region bounded by the line $y=2x+5$ and the parabola $y=x^2-2x$.
5. Find the area of the region bounded between the curves $y=\sin x$ and $y=\cos x$ and the lines $x=0$ and $x=\pi$.
6. Find the area of the region bounded by $y=\tan x$, $y=\cot x$ and the lines $x=0$, $x=\frac{\pi}{2}$, $y=0$.
7. Find the area of the region bounded by the parabola $y^2=x$ and the line $y=x-2$.

8. Father of a family wishes to divide his square field bounded by $x=0$, $x=4$, $y=4$ and $y=0$ along the curve $y^2=4x$ and $x^2=4y$ into three equal parts for his wife, daughter and son. Is it possible to divide? If so, find the area to be divided among them.
9. The curve $y=(x-2)^2+1$ has a minimum point at P. A point Q on the curve is such that the slope of PQ is 2. Find the area bounded by the curve and the chord PQ.
10. Find the area of the region common to the circle $x^2+y^2=16$ and the parabola $y^2=6x$.

9.9 Volume of a solid obtained by revolving area about an axis

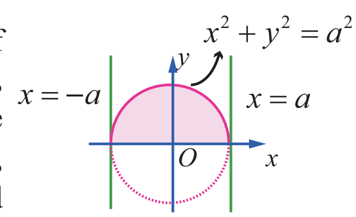


Fig. 9.34

Definite integrals have applications in finding volumes of solids of revolution about a fixed axis. By a solid of revolution about a fixed axis, we mean that a solid is generated when a plane region in a given plane undergoes one full revolution about a fixed axis in the plane. For instance, consider the semi circular plane region inside the circle $x^2+y^2=a^2$ and above the x-axis. See Fig.9.34.

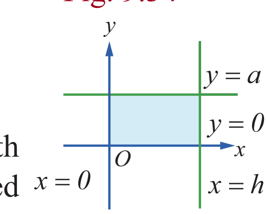


Fig. 9.35

If this region is given one complete rotation (revolution for $360°=2\pi$ radians) about x-axis, then a solid called a sphere is generated.

In the same manner, if you want to generate a right-circular cylinder with radius a and height h, you can consider the rectangular plane region bounded between the straight lines $y=0$, $y=a$, $x=0$ and $x=h$ in the xy-plane. See Fig.9.35. If this region is given one complete rotation (revolution for $360°=2\pi$ radians) about x-axis, then a solid called a cylinder is generated.

We restrict ourselves to obtain volume of solid of revolution about x-axis or y-axis. Whenever solid of revolution about x-axis is considered, the plane region that is revolved about x-axis lies above the x-axis. So, in this region $y \geq 0$. Whenever solid of revolution about y-axis is considered, the plane region that is revolved about y-axis lies to the right of y-axis. So, in this region $x \geq 0$. We shall find the formula for finding the volume of the solid of revolution of the plane region in the first quadrant bounded by the curve $y=f(x)$, x-axis and the lines $x=a$ and $x=b>a$ about x-axis. **The derivation of the formula is based upon the formula that the volume of a cylinder of radius r and the height h is $\pi r^2 h$.**

Assume that every line parallel to y-axis lying between the lines $x=a$ and $x=b>a$ cuts the curve $y=f(x)$ in the first quadrant exactly at one point. Divide $[a,b]$ into n segments by $x_1, x_2, \ldots, x_{n-1}$ such that

$$a=x_0<x_1<x_2<\ldots<x_{n-1}<x_n=b,\ x_i-x_{i-1}=\Delta x=\frac{b-a}{n},\ i=1,2,\ldots,n.$$

For each $i=0,1,2,\ldots,n-1$, the region in the xy-plane between the ordinates at x_i and $x_i+\Delta x$ which lies between the x-axis and the curve $y=f(x)$ can be **approximated** to an infinitesimal rectangle having area $y_i \Delta x$, where $y_i=f(x_i)$. When the plane region bounded by the curve $y=f(x)$, x-axis, and lines $x=a$ and $x=b$ is rotated by $360°$ about x-axis, each of the infinitesimal rectangles at $x=x_i$ also revolves and generates an elementary solid which is **approximately** a thin cylindrical disc with radius y_i and height Δx. See Fig.9.36. The volume of the cylindrical disc at

$x = x_i$ is given by $\pi y_i^2 \Delta x$, $i = 0,1,2,...,n-1$. Summing all these elementary volumes, we get the approximate volume of the solid of revolution as $\sum_{i=0}^{n-1} \pi y_i^2 \Delta x$. Let n become larger and larger $(n \to \infty)$ such that Δx becomes smaller and smaller $(\Delta x \to 0)$. Then $\sum_{i=0}^{n-1} \pi y_i^2 \Delta x$ tends to the volume of the solid of revolution. Hence the volume of the solid of revolution is $\pi \int_a^b y^2 dx$.

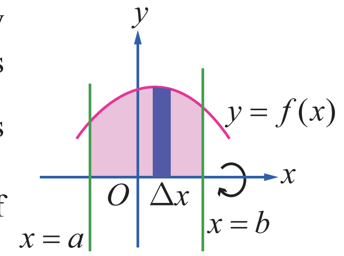


Fig. 9.36

Similarly, we can find the formula for finding the volume of the solid of revolution of the region bounded by the curve $x = f(y)$, y-axis, and the lines $y = c$ and $y = d$ about y-axis. The curve $x = f(y)$ lies to the right of y-axis between the lines $y = c$ and $y = d > c$. Assume that every line parallel to x-axis between $y = c$ and $y = d > c$ cuts the curve $x = f(y)$ in the first quadrant exactly at one point. See Fig.9.37. Then, the volume of the solid of revolution is given by $\pi \int_c^d x^2 dy$.

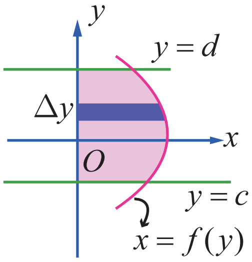


Fig. 9.37

Example 9.62

Find the volume of a sphere of radius a.

Solution

By revolving the upper semicircular region enclosed between the circle $x^2 + y^2 = a^2$ and the x-axis, we get a sphere of radius a. See Fig. 9.38.

The boundaries of the region are $y = \sqrt{a^2 - x^2}$, x-axis, the lines $x = -a$ and $x = a$. Hence, the volume of the sphere is given by

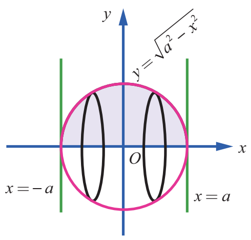


Fig. 9.38

$$V = \pi \int_{-a}^{a} y^2 dx = \pi \int_{-a}^{a} \left(a^2 - x^2\right) dx$$

$$= 2\pi \int_0^a \left(a^2 - x^2\right) dx \text{, since the integrand } \left(a^2 - x^2\right) \text{ is an even function.}$$

$$= 2\pi \left(a^2 x - \frac{x^3}{3}\right)_0^a = 2\pi \left(a^3 - \frac{a^3}{3}\right) = \frac{4}{3}\pi a^3 .$$

Example 9.63

Find the volume of a right-circular cone of base radius r and height h.

Solution

Consider the triangular region in the first quadrant which is bounded by the line $y = \frac{r}{h}x$, x-axis, the lines $x = 0$ and $x = h$. See Fig.9.39. By revolving the region about the x-axis, we get a cone of base radius r and height h.

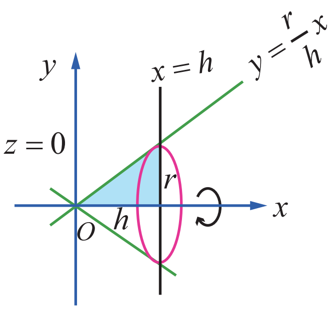


Fig. 9.39

Hence, the volume of the cone is given by

$$V = \pi \int_0^h y^2 dx = \pi \int_0^h \left(\frac{r}{h}x\right)^2 dx = \pi \left(\frac{r}{h}\right)^2 \int_0^h x^2 dx = \pi \left(\frac{r}{h}\right)^2 \left[\frac{x^3}{3}\right]_0^h = \frac{\pi r^2 h}{3}$$

Example 9.64

Find the volume of the spherical cap of height h cut of from a sphere of radius r.

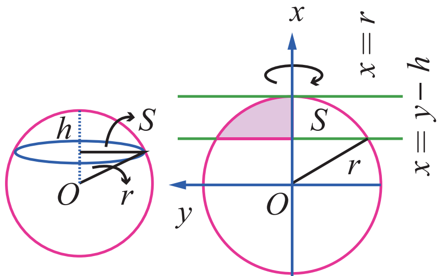


Fig. 9.40

Solution

If the region in the first quadrant bounded by the circle $x^2 + y^2 = r^2$, the x-axis, the lines $x = r - h$ and $x = r$ is revolved about the x-axis, then the solid generated is a spherical cap of height h cut of from a sphere of radius r. See Fig. 9.40. Hence, the required volume is given by

$$V = \pi\int_{r-h}^{r} y^2 dx = \pi\int_{r-h}^{r}\left(r^2 - x^2\right)dx = \pi\left(r^2 x - \frac{x^3}{3}\right)_{r-h}^{r}$$

$$= \pi\left(r^2\left(r-(r-h)\right) - \frac{\left(r^3 - (r-h)^3\right)}{3}\right) = \pi\left(r^2 h - \frac{\left(r^3 - \left(r^3 - 3r^2 h + 3rh^2 - h^3\right)\right)}{3}\right)$$

$$= \pi\left(\frac{3rh^2 - h^3}{3}\right) = \frac{1}{3}\pi h^2\left(3r - h\right).$$

■

Note

We can rewrite the above volume in terms of the radius of the cap.

If ρ is the radius of the cap, then $\rho^2 + (r-h)^2 = r^2$.

Then, we have $r = \dfrac{\rho^2 + h^2}{2h}$. Eliminating r, we get

$$V = \frac{1}{3}\pi h^2\left[3\left(\frac{\rho^2 + h^2}{2h}\right) - h\right] = \frac{1}{3}\pi h\left[\left(\frac{3\rho^2 + h^2}{2}\right)\right] = \frac{1}{6}\pi h\left(3\rho^2 + h^2\right).$$

Example 9.65

Find the volume of the solid formed by revolving the region bounded by the parabola $y = x^2$, x-axis, ordinates $x = 0$ and $x = 1$ about the x-axis.

Solution

The region to be revolved about the x-axis is sketched as in Fig.9.41. Hence, the required volume is given by

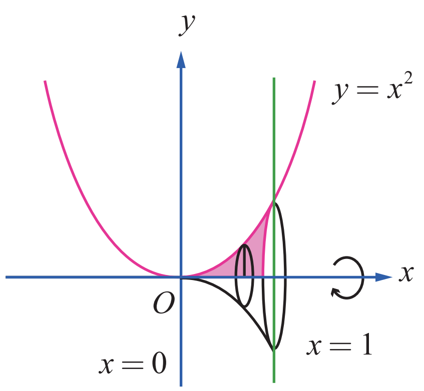


Fig. 9.41

$$V = \pi\int_0^1 y^2 dx$$

$$= \pi\int_0^1 \left(x^2\right)^2 dx$$

$$= \pi\left[\frac{x^5}{5}\right]_0^1$$

$$= \pi\left[\frac{1}{5} - 0\right] = \frac{\pi}{5}.$$

■

Example 9.66

Find the volume of the solid formed by revolving the region bounded by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1, a > b$ about the major axis.

Solution

The ellipse is symmetric about both the axes. The major axis lies along x-axis. The region to be revolved is sketched as in Fig.9.42.

Hence, the required volume is given by

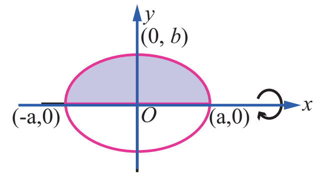


Fig. 9.42

$$V = \pi\int_{-a}^{a} y^2 dx = \pi\int_{-a}^{a} \frac{b^2}{a^2}\left(a^2 - x^2\right)dx$$

$$= \frac{2\pi b^2}{a^2}\int_{0}^{a}\left(a^2 - x^2\right)dx \text{, since the integrand is an even function.}$$

$$= \frac{2\pi b^2}{a^2}\left(a^2 x - \frac{x^3}{3}\right)_0^a = \frac{2\pi b^2}{a^2}\left(a^3 - \frac{a^3}{3}\right) = \frac{2\pi b^2}{a^2}\left(\frac{2a^3}{3}\right) = \frac{4\pi ab^2}{3}$$

Note

If the region bounded by ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is revolved about the y-axis, then the volume of the solid of revolution is $\frac{4\pi a^2 b}{3}$. The solid is called an ellipsoid.

Example 9.67

Find, by integration, the volume of the solid generated by revolving about y-axis the region bounded between the parabola $x = y^2 + 1$, the y-axis, and the lines $y = 1$ and $y = -1$.

Solution

The parabola $x = y^2 + 1$ is $y^2 = x - 1$. It is symmetrical about x-axis and has the vertex at $(1, 0)$ and focus at $\left(\frac{5}{4}, 0\right)$. The region for revolution is shaded in Fig.9.43. Hence, the required volume is given by

$$V = \pi\int_{-1}^{1} x^2 dy$$

$$= \pi\int_{-1}^{1}\left(y^2 + 1\right)^2 dy$$

$$= 2\pi\int_{0}^{1}\left(y^4 + 2y^2 + 1\right)dy \text{, since the integrand is an even function.}$$

$$= 2\pi\left(\frac{y^5}{5} + 2\frac{y^3}{3} + y\right)_0^1 = 2\pi\left(\frac{1}{5} + \frac{2}{3} + 1\right) = \frac{56}{15}\pi.$$

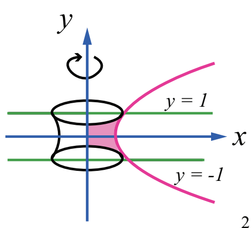


Fig. 9.43

Example 9.68

Find, by integration, the volume of the solid generated by revolving about y-axis the region bounded between the curve $y = \frac{3}{4}\sqrt{x^2 - 16}$, $x \geq 4$, the y-axis, and the lines $y = 1$ and $y = 6$.

Solution

We note that $y = \frac{3}{4}\sqrt{x^2 - 16} \Rightarrow \frac{x^2}{16} - \frac{y^2}{9} = 1$. So, the given curve is a portion of the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ between the lines $y = 1$ and $y = 6$ and it lies above the x-axis.

The region to be revolved is sketched in Fig.9.44.

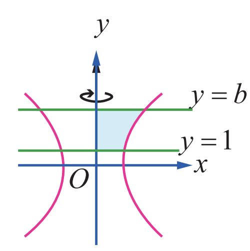


Fig. 9.44

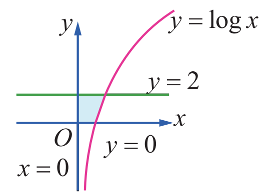
Since revolution is made about y-axis, we write the equation of the portion of the hyperbola as $x = \frac{4}{3}\sqrt{9+y^2}$. So, the volume of the solid generated is given by

$$V = \pi\int_1^6 x^2 dy = \pi\int_1^6 \left(\frac{4}{3}\sqrt{9+y^2}\right)^2 dy = \pi\left(\frac{16}{9}\right)\int_1^6 \left(9+y^2\right) dy$$

$$= \pi\left(\frac{16}{9}\right)\left(9y+\frac{y^3}{3}\right)_1^6 = \pi\left(\frac{16}{9}\right)\left[(54+72)-(9+\frac{1}{3})\right] = \frac{5600}{27}\pi$$

Example 9.69

Find, by integration, the volume of the solid generated by revolving about y-axis the region bounded by the curves $y = \log x$, $y = 0$, $x = 0$ and $y = 2$.



Solution

The region to be revolved is sketched in Fig.9.45.

Fig. 9.45

Since revolution is made about the y-axis, the volume of the solid generated is given by

$$V = \pi\int_0^2 x^2 dy = \pi\int_0^2 \left(e^y\right)^2 dy = \pi\int_0^2 e^{2y} dy$$

$$= \pi\left[\frac{e^{2y}}{2}\right]_0^2 = \frac{\pi}{2}\left(e^4-1\right).$$

EXERCISE 9.9

1. Find, by integration, the volume of the solid generated by revolving about the x-axis, the region enclosed by $y = 2x^2$, $y = 0$ and $x = 1$.
2. Find, by integration, the volume of the solid generated by revolving about the x-axis, the region enclosed by $y = e^{-2x}$ $y = 0$, $x = 0$ and $x = 1$
3. Find, by integration, the volume of the solid generated by revolving about the y-axis, the region enclosed by $x^2 = 1+y$ and $y = 3$.
4. The region enclosed between the graphs of $y = x$ and $y = x^2$ is denoted by R, Find the volume generated when R is rotated through $360°$ about x-axis.
5. Find, by integration, the volume of the container which is in the shape of a right circular conical frustum as shown in the Fig 9.46.
6. A watermelon has an ellipsoid shape which can be obtained by revolving an ellipse with major-axis 20 cm and minor-axis 10 cm about its major-axis. Find its volume using integration.

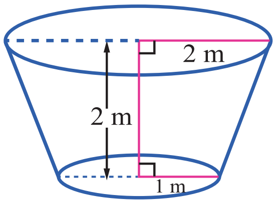


Fig. 9.46

EXERCISE 9.10

Choose the correct or the most suitable answer from the given four alternatives :

1. The value of $\int_0^{\frac{2}{3}} \frac{dx}{\sqrt{4-9x^2}}$ is

(1) $\frac{\pi}{6}$ (2) $\frac{\pi}{2}$ (3) $\frac{\pi}{4}$ (4) π

X1B2J2

2. The value of $\int_{-1}^{2} |x|\, dx$ is

(1) $\frac{1}{2}$ (2) $\frac{3}{2}$ (3) $\frac{5}{2}$ (4) $\frac{7}{2}$

3. For any value of $n \in \mathbb{Z}$, $\int_0^{\pi} e^{\cos^2 x} \cos^3 [(2n+1)x]\, dx$ is

(1) $\frac{\pi}{2}$ (2) π (3) 0 (4) 2

4. The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \cos x\, dx$ is

(1) $\frac{3}{2}$ (2) $\frac{1}{2}$ (3) 0 (4) $\frac{2}{3}$

5. The value of $\int_{-4}^{4} \left[\tan^{-1}\left(\frac{x^2}{x^4+1}\right) + \tan^{-1}\left(\frac{x^4+1}{x^2}\right) \right] dx$ is

(1) π (2) 2π (3) 3π (4) 4π

6. The value of $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\frac{2x^7 - 3x^5 + 7x^3 - x + 1}{\cos^2 x} \right) dx$ is

(1) 4 (2) 3 (3) 2 (4) 0

7. If $f(x) = \int_0^x t \cos t\, dt$, then $\frac{df}{dx} =$

(1) $\cos x - x\sin x$ (2) $\sin x + x\cos x$ (3) $x\cos x$ (4) $x\sin x$

8. The area between $y^2 = 4x$ and its latus rectum is

(1) $\frac{2}{3}$ (2) $\frac{4}{3}$ (3) $\frac{8}{3}$ (4) $\frac{5}{3}$

9. The value of $\int_0^1 x(1-x)^{99}\, dx$ is

(1) $\frac{1}{11000}$ (2) $\frac{1}{10100}$ (3) $\frac{1}{10010}$ (4) $\frac{1}{10001}$

10. The value of $\int_0^{\pi} \frac{dx}{1+5^{\cos x}}$ is

(1) $\frac{\pi}{2}$ (2) π (3) $\frac{3\pi}{2}$ (4) 2π

11. If $\frac{\Gamma(n+2)}{\Gamma(n)} = 90$ then n is

(1) 10 (2) 5 (3) 8 (4) 9

12. The value of $\int_0^{\frac{\pi}{6}} \cos^3 3x\, dx$ is

(1) $\frac{2}{3}$ (2) $\frac{2}{9}$ (3) $\frac{1}{9}$ (4) $\frac{1}{3}$

13. The value of $\int_0^{\pi} \sin^4 x \, dx$ is

(1) $\frac{3\pi}{10}$ (2) $\frac{3\pi}{8}$ (3) $\frac{3\pi}{4}$ (4) $\frac{3\pi}{2}$

14. The value of $\int_0^{\infty} e^{-3x} x^2 dx$ is

(1) $\frac{7}{27}$ (2) $\frac{5}{27}$ (3) $\frac{4}{27}$ (4) $\frac{2}{27}$

15. If $\int_0^{a} \frac{1}{4+x^2} dx = \frac{\pi}{8}$ then a is

(1) 4 (2) 1 (3) 3 (4) 2

16. The volume of solid of revolution of the region bounded by $y^2 = x(a-x)$ about x-axis is

(1) πa^3 (2) $\frac{\pi a^3}{4}$ (3) $\frac{\pi a^3}{5}$ (4) $\frac{\pi a^3}{6}$

17. If $f(x) = \int_1^{x} \frac{e^{\sin u}}{u} du, x > 1$ and

$\int_1^{3} \frac{e^{\sin x^2}}{x} dx = \frac{1}{2}[f(a) - f(1)]$, then one of the possible value of a is

(1) 3 (2) 6 (3) 9 (5)

18. The value of $\int_0^{1} \left(\sin^{-1} x\right)^2 dx$ is

(1) $\frac{\pi^2}{4} - 1$ (2) $\frac{\pi^2}{4} + 2$ (3) $\frac{\pi^2}{4} + 1$ (4) $\frac{\pi^2}{4} - 2$

19. The value of $\int_0^{a} \left(\sqrt{a^2 - x^2}\right)^3 dx$ is

(1) $\frac{\pi a^3}{16}$ (2) $\frac{3\pi a^4}{16}$ (3) $\frac{3\pi a^2}{8}$ (4) $\frac{3\pi a^4}{8}$

20. If $\int_0^{x} f(t)\, dt = x + \int_x^{1} t f(t)\, dt$, then the value of $f(1)$ is

(1) $\frac{1}{2}$ (2) 2 (3) 1 (4) $\frac{3}{4}$

SUMMARY

(1) Definite integral as the limit of a sum

(i) $$\int_a^b f(x)dx = \lim_{n\to\infty}\frac{b-a}{n}\sum_{r=1}^{n} f\left(a+(b-a)\frac{r}{n}\right)$$

(ii) $$\int_0^1 f(x)dx = \lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n} f\left(\frac{r}{n}\right) = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right).$$

(2) Properties of definite integrals

(i) $$\int_a^b f(x)\,dx = \int_a^b f(u)\,du$$

(ii) $$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

(iii) $$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

(iv) $$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$$

(v) $$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

(vi) $$\int_0^{2a} f(x)\,dx = \int_0^a [f(x)+f(2a-x)]\,dx.$$

(vii) If $f(x)$ is an even function, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx.$

(ix) If $f(x)$ is an odd function, then $\int_{-a}^{a} f(x)\,dx = 0.$

(x) If $f(2a-x) = f(x)$, then $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx.$

(xi) If $f(2a-x) = -f(x)$, then $\int_0^{2a} f(x)\,dx = 0.$

(xii) $$\int_0^a x\,f(x)\,dx = \frac{a}{2}\int_0^a f(x)\,dx \text{ if } f(a-x) = f(x)$$

(3) Bernoulli's Formula

$$\int uv dx = uv_{(1)} - u^{(1)}v_{(2)} + u^{(2)}v_{(3)} - u^{(3)}v_{(4)} + \cdots.$$

(4) Reduction Formulae

(i) $$\int_0^{\frac{\pi}{2}} \sin^n x\;dx = \int_0^{\frac{\pi}{2}} \cos^n x\;dx = \begin{cases} \dfrac{(n-1)}{n}\times\dfrac{(n-3)}{(n-2)}\times\cdots\times\dfrac{1}{2}\times\dfrac{\pi}{2}, & \text{if } n = 2,4,6,\cdots \\ \dfrac{(n-1)}{n}\times\dfrac{(n-3)}{(n-2)}\times\cdots\times\dfrac{2}{3}, & \text{if } n = 3,5,7,\cdots \end{cases}$$

(ii) If n is even and m is even,

$$\int_0^{\frac{\pi}{2}} \sin^m x\cos^n x\;dx = \frac{(n-1)}{(m+n)}\frac{(n-3)}{(m+n-2)}\frac{(n-5)}{(m+n-4)}\cdots\frac{1}{(m+2)}\frac{(m-1)}{m}\frac{(m-3)}{(m-2)}\frac{(m-5)}{(m-4)}\cdots\frac{1}{2}\frac{\pi}{2}$$

(iii) If n is odd and m is any positive integer (even or odd), then

$$\int_0^{\frac{\pi}{2}} \sin^m x\;\cos^n x\;dx = \frac{(n-1)}{(m+n)}\frac{(n-3)}{(m+n-2)}\frac{(n-5)}{(m+n-4)}\cdots\frac{2}{(m+3)}\frac{1}{(m+1)}.$$

(5) Gamma Formulae

(i) $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx = (n-1)!$ (ii) $\int_0^\infty e^{-ax} x^n dx = \dfrac{n!}{a^{n+1}}$

(6) Area of the region bounded by a curve and lines

(i) The area of the region bounded by a curve, above x-axis and the lines $x = a$ and $x = b$ is $A = \int_a^b y dx$.

(ii) The area of the region bounded by a curve, below x-axis and the lines $x = a$ and $x = b$ is $A = -\int_a^b y dx = \left|\int_a^b y dx\right|$.

(iii) Thus area of the region bounded by the curve to the right of y-axis, the lines $y = c$ and $y = d$ is $A = \int_c^d x dy$.

(iv) The area of the region bounded by the curve to the left of y-axis, the lines $y = c$ and $y = d$ is $A = -\int_c^d x dy = \left|\int_c^d x dy\right|$.

(v) If $f(x) \geq g(x)$, then area bounded by the curves and the lines $x = a$, $x = b$ is

$$A = \int_a^b [f(x) - g(x)] dx = \int_a^b (y_U - y_L) dx$$

(vi) If $f(y) \geq g(y)$, then area bounded by the curves and the lines $y = c$, $y = d$ is

$$A = \int_c^d [f(y) - g(y)] dy = \int_c^d (x_R - x_L) dy$$

(7) Volume of the solid of revolution

(i) The volume of the solid of revolution about x-axis is $V = \pi\int_a^b y^2 dx$.

(ii) The volume of the solid of revolution about y-axis is $V = \pi\int_c^d x^2 dy$.

ICT CORNER

https://ggbm.at/dy9kwgbt or Scan the QR Code

Open the Browser, type the URL Link given (or) Scan the QR code. GeoGebra work book named "**12th Standard Mathematics Vol-2**" will open. In the left side of work book there are chapters related to your text book. Click on the chapter named "**Applications of Integration**". You can see several work sheets related to the chapter. Go through all the work sheets

B262_12_MATHS_EM

Chapter 10

Ordinary Differential Equations

X3E4X8

"Mathematics is the most beautiful and most powerful creation of the human spirit"

- Stefan Banach

10.1 Introduction

Motivation and Early Developments

Just we look at some real life situations where

- the motion of projectile, rocket, satellite and planets
- the charge or current in the electric circuit
- the conduction of heat on a rod or in a slab
- the vibrations of a wire or membrane etc

are to be determined. The mathematical formulations of such problems emerge as differential equations under certain scientific laws. These laws involve various rates of change (derivatives) of one or more quantities with respect to other quantities. Thus the scientific laws manifest as mathematical equations involving derivatives, viz. differential equations.

Differential Equations emanate from the problems in geometry, mechanics, physics, chemistry, and engineering studies. We have studied about "rates" in our early classes. This is also known as instantaneous rate of change which is denoted as $\frac{dy}{dx}$.

We give below some relations between the rate of change and unknown functions that occur in real life situations.

(a) The rate of change of y with respect to x is directly proportional to y:

$$\frac{dy}{dx} = ky.$$

(b) The rate of change of y with respect to x is directly proportional to the product of y^2 and x :

$$\frac{dy}{dx} = ky^2x.$$

(c) The rate of change of y with respect to x is inversely proportional to y:

$$\frac{dy}{dx} = \frac{k}{y}.$$

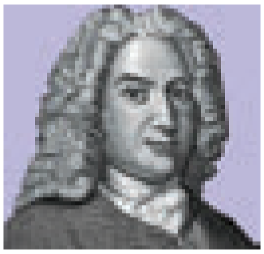
(d) The rate of change of y with respect to x is directly proportional to y^2 and inversely proportional to $\sqrt{x}$:

$$\frac{dy}{dx} = k\frac{y^2}{\sqrt{x}}.$$

A differential equation is an equation in which some derivatives of the unknown function occur. In many cases the independent variable is taken to be time.

In order to apply mathematical methods to a physical or "real life" problem, we must formulate the problem in mathematical terms; that is, we must construct a mathematical model for the problem. Many physical problems concern relationships between changing quantities. Since rates of change are represented mathematically by derivatives, mathematical models often involve equations relating to an unknown function and one or more of its derivatives. Such equations are differential equations. They are of basic significance in science and engineering since many physical laws as well as relations are modelled in the form of differential equations. Differential equations are much useful in describing mathematical models involving population growth or radio-active decay. The study of biological sciences and economics is incomplete without the application of differential equations.

The subject of differential equations was invented along with calculus by Newton and Leibniz in order to solve problems in geometry and physics. It played a crucial part in the development of Newtonian physics by the Bernoulli family, Euler, and others. Some of the applications of differential equations in our daily life are found in mobile phones, motor cars, air flights, weather forecast, internet, health care, or in many other daily activities.


Johann Bernoulli
(1667-1748)

In this chapter, we introduce and discuss the first order ordinary differential equations and some methods to find their solutions.

Learning Objectives

Upon completion of this chapter, students will be able to

- classify differential equations
- construct differential equations
- find the order and degree of the differential equations
- solve differential equation using the methods of variables separable, substitution, integrating factor
- apply differential equation in real life problems

10.2 Differential Equation, Order, and Degree

Definition 10.1

A **differential equation** is any equation which contains at least one derivative of an unknown function, either ordinary derivative or partial derivative.

For instance, let $y = f(x)$ where y is a dependent variable (f is an unknown function) and x is an independent variable.

(1) The equation $\frac{dy}{dx} = 0$ is a differential equation.

(2) The equation $\frac{dy}{dx} = \sin x$ is a differential equation.

(3) The equation $\frac{dy}{dx} + y = 7x + 5$ is a differential equation.

(4) The equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \sin x$ is a differential equation.

(5) The equation $e^{\frac{dy}{dx}} = \ln x, x > 0$ is a differential equation.

(6) The equation $\tan^{-1}\left(\frac{d^2y}{dx^2} + y^2 + 2x\right) = \frac{dy}{dx}$ is a differential equation.

Definition 10.2 (Order of a differential equation)

The **order** of a differential equation is the highest order derivative present in the differential equation.

Thus, if the highest order derivative of the unknown function y in the equation is k^{th} derivative, then the order of the differential equation is k. Clearly k must be a positive integer.

For example, $\left(\frac{d^3y}{dx^3}\right)^{\frac{2}{3}} - 3\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 4 = 0$ is a differential equation of order three.

Definition 10.3 (Degree of a differential equation)

If a differential equation is expressible in a polynomial form, then the integral power of the highest order derivative appears is called the **degree** of the differential equation

In other words, the **degree** of a differential equation is the power of the highest order derivative involved in the differential equation when the differential equation (after expressing in polynomial form) satisfies the following conditions :

(i) All of the derivatives in the equation are free from fractional powers, if any.

(ii) Highest order derivative should not be an argument of a transcendental function, trigonometric or exponential, etc. The coefficient of any term containing the highest order derivative should just be a function of *x, y*, or some lower order derivative but not as transcendental, trigonometric, exponential, logarithmic function of derivatives.

If one or more of the aforementioned conditions are not satisfied by the differential equation, it should be first reduced to the polynomial form in which it satisfies all of the above conditions.

If a differential equation is not expressible to polynomial equation form having the highest order derivative as the leading term then that the degree of the differential equation is not defined.

The determination of the degree of a given differential equation can be tricky if you are not well versed with the conditions under which the degree of the differential equation is defined. So go through the given solved examples carefully and master the technique of calculating the degree of the given differential equation just by sheer inspection!

Examples for the calculation of degree:

(1) Consider the differential equation $3y^2\left(\frac{dy}{dx}\right)^3 - \frac{d^2y}{dx^2} = \sin x^2$.

The highest order derivative involved here is 2, and its power is 1 in the equation. Thus, the order of the differential equation is 2 and degree is 1.

(2) Consider the differential equation $\sqrt{1+\left(\frac{dy}{dx}\right)^2} = y\frac{d^3y}{dx^3}$.

Since this equation involves fractional powers, we must first get rid of them. On squaring the equation, we get

$$1+\left(\frac{dy}{dx}\right)^2 = y^2\left(\frac{d^3y}{dx^3}\right)^2.$$

Now, we can clearly make out that the highest order derivative is 3. Therefore order of the differential equation is 3 and since its power is 2 in the equation, the degree of the differential equation is 2.

(3) Consider the differential equation $\sin\left(\frac{dy}{dx}\right)+\frac{d^2y}{dx^2}+3x=0$.

Here, the highest order derivative is 2. Because of sine of first derivative, the given differential equation can not be expressed as polynominal equation. So, the order of the differential equation is 2, and, it is not in polynomial equation in derivatives and so degree is not defined.

(4) Consider the equation $e^{\frac{d^2y}{dx^2}}+\sin(x)\frac{dy}{dx}=2$.

Here, the highest order derivative (order is 2) has involvement in an exponential function. This cannot be expressed as polynomial equation with $\frac{d^2y}{dx^2}$ as the leading term So, the degree of the equation is not defined. The order of the equation is 2.

(5) Further, the following differential equations do not have degrees.

(i) $e^{\frac{dy}{dx}}+\frac{dy}{dx}=0$ (ii) $\log\left(\frac{d^2y}{dx^2}\right)+\frac{dy}{dx}=0$ and (iii) $\cos\left(\frac{d^3y}{dx^3}\right)+2\frac{d^2y}{dx^2}=0$.

(6) The differential equation $10(y''')^4+7(y'')^5+\sin(y')+5=0$ has order 3 but degree is not defined.

(7) The differential equation $\cos(y')y'''+5y''+7y'=\sin x$ has order 3 and degree is not defined.

Remark

Observe that the degree of a differential equation is always a positive integer.

Example 10.1

Determine the order and degree (if exists) of the following differential equations:

(i) $\frac{dy}{dx}=x+y+5$ (ii) $\left(\frac{d^4y}{dx^4}\right)^3+4\left(\frac{dy}{dx}\right)^7+6y=5\cos 3x$

(iii) $\frac{d^2y}{dx^2}+3\left(\frac{dy}{dx}\right)^2=x^2\log\left(\frac{d^2y}{dx^2}\right)$ (iv) $3\left(\frac{d^2y}{dx^2}\right)=\left[4+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}$

(v) $dy+\left(xy-\cos x\right)dx=0$

Solution

(i) In this equation, the highest order derivative is $\frac{dy}{dx}$ whose power is 1

Therefore, the given differential equation is of order 1 and degree 1.

(ii) Here, the highest order derivative is $\frac{d^4y}{dx^4}$ whose power is 3.

Therefore, the given differential equation is of order 4 and degree 3.

(iii) In the given differential equation, the highest order derivative is $\frac{d^2y}{dx^2}$ whose power is 1.

Therefore, the given differential equation is of order 2.

The given differential equation is not a polynomial equation in its derivatives and so its degree is not defined.

(iv) The given differential equation is $3\left(\frac{d^2y}{dx^2}\right)=\left[4+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}$

Squaring both sides, we get $9\left(\frac{d^2y}{dx^2}\right)^2=\left[4+\left(\frac{dy}{dx}\right)^2\right]^3$.

In this equation, the highest order derivative is $\frac{d^2y}{dx^2}$ whose power is 2.

Therefore, the given differential equation is of order 2 and degree 2.

(v) $dy+(xy-\cos x)dx=0$ is a first order differential equation with degree 1, since the equation can be rewritten as $\frac{dy}{dx}+xy-\cos x=0$.

EXERCISE 10.1

1. For each of the following differential equations, determine its order, degree (if exists)

(i) $\frac{dy}{dx}+xy=\cot x$

(ii) $\left(\frac{d^3y}{dx^3}\right)^{\frac{2}{3}}-3\frac{d^2y}{dx^2}+5\frac{dy}{dx}+4=0$

(iii) $\left(\frac{d^2y}{dx^2}\right)^2+\left(\frac{dy}{dx}\right)^2=x\sin\left(\frac{d^2y}{dx^2}\right)$

(iv) $\sqrt{\frac{dy}{dx}}-4\frac{dy}{dx}-7x=0$

(v) $y\left(\frac{dy}{dx}\right)=\frac{x}{\left(\frac{dy}{dx}\right)+\left(\frac{dy}{dx}\right)^3}$

(vi) $x^2\frac{d^2y}{dx^2}+\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}=0$

(vii) $\left(\frac{d^2y}{dx^2}\right)^3=\sqrt{1+\left(\frac{dy}{dx}\right)}$

(viii) $\frac{d^2y}{dx^2}=xy+\cos\left(\frac{dy}{dx}\right)$

(ix) $\frac{d^2y}{dx^2}+5\frac{dy}{dx}+\int y\,dx=x^3$

(x) $x=e^{xy\left(\frac{dy}{dx}\right)}$

10.3 Classification of Differential Equations

Definition 10.4: (Ordinary Differential Equation)

If a differential equation contains only ordinary derivatives of one or more functions with respect to a single independent variable, it is said to be an **Ordinary Differential Equation (ODE)**.

Definition 10.5: (Partial Differential Equation)

An equation involving only partial derivatives of one or more functions of two or more independent variables is called a **Partial Differential Equation (PDE)**.

For instance, let y denote the unknown function and x be independent variable. Then $\frac{dy}{dx}+2y=e^{-x}$, $\frac{d^2y}{dx^2}-\frac{dy}{dx}-5y=0$ and $\frac{dx}{dt}+\frac{dy}{dt}=3x-4y$ are some examples of ordinary differential equations.

For instance, $\frac{\partial u}{\partial y}=-\frac{\partial u}{\partial x}$, $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ and $\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}-2\frac{\partial u}{\partial t}$ are some examples of partial differential equations.

In this chapter, we discuss ordinary differential equations only.

Ordinary differential equations are classified into two different categories namely **linear** ordinary differential equations and **nonlinear** ordinary differential equations.

Definition 10.6

A general linear ordinary differential equation of order *n* is any differential equation that can be written in the following form.

$$a_n(x)y^{(n)}+a_{n-1}(x)y^{(n-1)}+\cdots+a_1(x)y'+a_0y=g(x) \quad \text{... (1)}$$

where the coefficients $a_n(x)\neq 0, a_0(x), a_1(x), \ldots, a_{n-1}(x)$ and $g(x)$ are any function of independent variable x (including the zero function)

Note

(1) The important thing to note about linear differential equations is that there are no products of the function, $y(x)$, and its derivatives and neither the function nor its derivatives occur to any power other than the first power.

(2) No transcendental functions – (trigonometric or logarithmic etc) of y or any of its derivatives occur in differential equation.

(3) Also note that neither the function nor its derivatives are "inside" another function, for instance, $\sqrt{y'}$ or $e^{y'}$.

(4) The coefficients $a_0(x), a_1(x), \ldots, a_{n-1}(x)$ and $g(x)$ can be zero or non-zero functions, or constant or non-constant functions, linear or non-linear functions. Only the function, $y(x)$, and its derivatives are used in determining whether a differential equation is linear.

Definition 10.7

A **nonlinear ordinary differential equation** is simply one that is not linear.

If the coefficients of $y, y', y'', \ldots, y^{(n)}$ contain the dependent variable y or its derivatives or if powers of $y, y', y'', \ldots, y^{(n)}$, such as $(y')^2$, appear in the equation, then the differential equation is nonlinear. Also, nonlinear functions of the dependent variable or its derivatives, such as $\sin y$ or $e^{y'}$ cannot appear in a linear equation.

For instance,

(1) $\dfrac{dy}{dx} = ax^3$, $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = 0$ and $\dfrac{dy}{dx} + p(x)y = q(x)$ are linear differential equations

where as $y\dfrac{dy}{dx} + \sin x = 0$ is a nonlinear differential equation.

(2) $y'' + 2x^3 y' = 7xy + x^2$ is a second order linear ODE.

(3) $y'' + y' = \sqrt{x}$ is a second order linear ODE.

(4) $y^2 + y' = \sqrt{x}$ is a first order nonlinear ODE.

(5) $y' = x\sin(y)$ is a first order nonlinear ODE.

(6) $y'' = y\sin(x)$ is a second order linear ODE.

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Definition 10.8

If $g(x) = 0$ in (1), then the above equation is said to be **homogeneous**, otherwise it is called **non-homogeneous**.

Remark

If $y_i(x),\ i = 1, 2$ are any two solutions of homogeneous equation

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = 0 \qquad \ldots(2)$$

then $a_n(x)y_i^{(n)}(x) + a_{n-1}(x)y_i^{(n-1)}(x) + \cdots + a_1(x)y_i(x) + a_0(x)y_i(x) = 0,\ i = 1, 2$.

Suppose $u(x) = c_1y_1(x) + c_2y_2(x)$, where c_1 and c_2 are arbitrary constants. Then, it can be easily verified that $u(x)$ is also a solution of (2).

Thus, a first order linear differential equation is written as $y' + p(x)y = f(x)$. A first order differential equation that can't be written like this is nonlinear. Since $y = 0$ is obviously a solution of the homogeneous equation $y' + p(x)y = 0$, we call it the trivial solution. Any other solution is nontrivial. In fact this is true for a general linear homogeneous differential equation as well.

10.4. Formation of Differential Equations

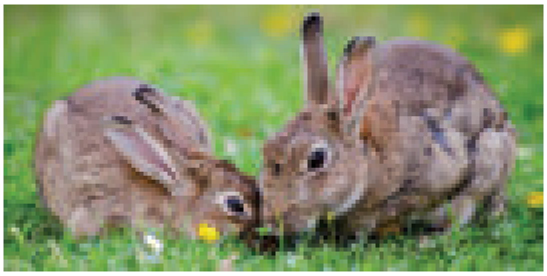
10.4.1 Formation of Differential equations from Physical Situations

Now, we provide some models to describe how the differential equations arise as models of real life problems.

Model 1: (Newton's Law)

According to Newton's second law of motion, the instantaneous acceleration a of an object with constant mass m is related to the force F acting on the object by the equation $F = ma$. In the case of a free fall, an object is released from a height $h(t)$ above the ground level. Then, the Newton's second law is described by the differential equation $m\frac{d^2h}{dt^2} = f\left(t, h(t), \frac{dh}{dt}\right)$, where m is the mass of the object, h is the height above the ground level. This is the second order differential equation of the unknown height as a function of time.

Model 2: (Population Growth Model)



The population will increase whenever the offspring increase. For instance, let us take rabbits as our population. More number of rabbits yield more number of baby rabbits. As time increases the population of rabbits increases. If the rate of growth of biomass $N(t)$ of the population at time t is proportional to the biomass of the population, then the differential equation governing the population is given by $\frac{dN}{dt} = rN$, where $r > 0$ is the growth rate.

Model 3: (Logistic Growth Model)

The rate at which a disease is spread (*i.e.,* the rate of increase of the number N of people infected) in a fixed population L is proportional to the product of the number of people infected and the number of people not yet infected:

$$\frac{dN}{dL} = kN(L - N), \ k > 0.$$

EXERCISE 10.2

1. Express each of the following physical statements in the form of differential equation.
   (i) Radium decays at a rate proportional to the amount Q present.
   (ii) The population P of a city increases at a rate proportional to the product of population and to the difference between 5,00,000 and the population.
   (iii) For a certain substance, the rate of change of vapor pressure P with respect to temperature T is proportional to the vapor pressure and inversely proportional to the square of the temperature.

(iv) A saving amount pays 8% interest per year, compounded continuously. In addition, the income from another investment is credited to the amount continuously at the rate of ₹ 400 per year.

2. Assume that a spherical rain drop evaporates at a rate proportional to its surface area. Form a differential equation involving the rate of change of the radius of the rain drop.

10.4.2 Formation of Differential Equations from Geometrical Problems

Given a family of functions parameterized by some constants, a differential equation can be formed by eliminating those constants of this family. For instance, the elimination of constants A and B from $y = \mathrm{A}e^{x} + \mathrm{B}e^{-x}$, yields a differential equation $\frac{d^2y}{dx^2} - y = 0$.

Consider an equation of a family of curves, which contains n arbitrary constants. To form a differential equation not containing any of these constants, let us proceed as follows:

Differentiate the given equation successively n times, getting n differential equations. Then eliminate n arbitrary constants from $(n+1)$ equations made up of the given equation and n newly obtained equations arising from n successive differentiations. The result of elimination gives the required differential equation which must contain a derivative of the nth order.

Example 10.2

Find the differential equation for the family of all straight lines passing through the origin.

Solution

The family of straight lines passing through the origin is $y = mx$, where m is an arbitrary constant. ... (1)

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = m. \qquad \text{... (2)}$$

From (1) and (2), we get $y = x\frac{dy}{dx}$. This is the required differential equation.

Observe that the given equation $y = mx$ contains only one arbitrary constant and thus we get the differential equation of order one.

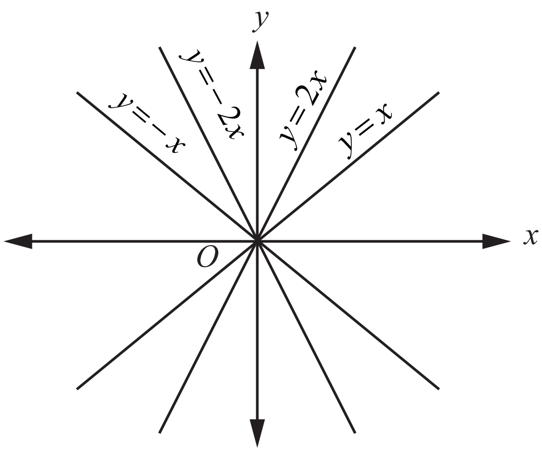


Fig. 10.1

Example 10.3

Form the differential equation by eliminating the arbitrary constants A and B from $y = \mathrm{A}\cos x + \mathrm{B}\sin x$.

Solution

Given that $y = \mathrm{A}\cos x + \mathrm{B}\sin x$ ... (1)

Differentiating (1) twice successively, we get

$$\frac{dy}{dx} = -\mathrm{A}\sin x + \mathrm{B}\cos x. \qquad \text{... (2)}$$

$$\frac{d^2y}{dx^2} = -\mathrm{A}\cos x - \mathrm{B}\sin x = -(\mathrm{A}\cos x + \mathrm{B}\sin x). \qquad \text{... (3)}$$

Substituting (1) in (3), we get $\frac{d^2y}{dx^2}+y=0$ as the required differential equation. ■

Example 10.4

Find the differential equation of the family of circles passing through the points $(a,0)$ and $(-a,0)$.

Solution

A circle passing through the points $(a,0)$ and $(-a,0)$ has its centre on y - axis.

Let $(0,b)$ be the centre of the circle. So, the radius of the circle is $\sqrt{a^2+b^2}$.

Therefore the equation of the family of circles passing through the points $(a,0)$ and $(-a,0)$ is $x^2+(y-b)^2=a^2+b^2$, b is an arbitrary constant. ... (1)

Differentiating both sides of (1) with respect to x, we get

$$2x+2(y-b)\frac{dy}{dx} = 0 \Rightarrow y-b=-\frac{x}{\frac{dy}{dx}} \Rightarrow b=\frac{x}{\frac{dy}{dx}}+y.$$

Substituting the value of b in equation (1), we get

$$x^2+\frac{x^2}{\left(\frac{dy}{dx}\right)^2} = a^2+\left(\frac{x}{\frac{dy}{dx}}+y\right)^2$$

$$\Rightarrow x^2\left(\frac{dy}{dx}\right)^2+x^2 = a^2\left(\frac{dy}{dx}\right)^2+\left[x+y\left(\frac{dy}{dx}\right)^2\right]^2$$

$$\Rightarrow (x^2-y^2-a^2)\frac{dy}{dx}-2xy=0$$, which is the required differential equation. ■

Example 10.5

Find the differential equation of the family of parabolas $y^2=4ax$, where a is an arbitrary constant.

Solution

The equation of the family of parabolas is given by $y^2=4ax$, a is an arbitrary constant. ... (1)

Differentiating both sides of (1) with respect to x, we get $2y\frac{dy}{dx}=4a \quad \Rightarrow \quad a=\frac{y}{2}\frac{dy}{dx}$

Substituting the value of a in (1) and simplifying, we get $\frac{dy}{dx}=\frac{y}{2x}$ as the required differential equation. ■

Example 10.6

Find the differential equation of the family of all ellipses having foci on the x-axis and centre at the origin.

Solution

The equation of the family of all ellipses having foci on the x-axis and centre at the origin is given by $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad a>b$ ... (1)

where a and b are arbitrary constants.

Differentiating equation (1) with respect to x, we get

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0 \Rightarrow \frac{x}{a^2}+\frac{y}{b^2}\frac{dy}{dx}=0 \qquad \text{... (2)}$$

Differentiating equation (2) with respect to x, we get

$$\frac{1}{a^2}+\frac{1}{b^2}\left[y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right]=0 \Rightarrow \frac{1}{a^2}=-\frac{1}{b^2}\left[y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right]$$

Substituting the value of $\frac{1}{a^2}$ in equation (2) and simplifying, we get

$$-\frac{1}{b^2}\left[y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right]x+\frac{y}{b^2}\frac{dy}{dx}=0 \Rightarrow xy\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2-y\frac{dy}{dx}=0$$

which is the required differential equation. ■

Remark

The result of eliminating one arbitrary constant yields a first order differential equation and that of eliminating two arbitrary constants leads to a second order differential equation and so on.

EXERCISE 10.3

1. Find the differential equation of the family of (i) all non-vertical lines in a plane (ii) all non-horizontal lines in a plane.
2. Form the differential equation of all straight lines touching the circle $x^2+y^2=r^2$.
3. Find the differential equation of the family of circles passing through the origin and having their centres on the x-axis.
4. Find the differential equation of the family of all the parabolas with latus rectum $4a$ and whose axes are parallel to the x-axis.
5. Find the differential equation of the family of parabolas with vertex at $(0,-1)$ and having axis along the y-axis.
6. Find the differential equations of the family of all the ellipses having foci on the y-axis and centre at the origin.
7. Find the differential equation corresponding to the family of curves represented by the equation $y=\mathrm{A}e^{8x}+\mathrm{B}e^{-8x}$, where A and B are arbitrary constants.
8. Find the differential equation of the curve represented by $xy=ae^x+be^{-x}+x^2$.

10.5 Solution of Ordinary Differential Equations

Definition 10.9 : (Solution of DE)

A **solution** of a differential equation is an expression for the dependent variable in terms of the independent variable(s) which satisfies the differential equation.

Caution

(i) There is no guarantee that a differential equation has a solution.

For instance, $\left(y'(x)\right)^2 + y^2 + 1 = 0$ has no solution, since $\left(y'(x)\right)^2 = -\left(y^2+1\right)$ and so $y'(x)$ cannot be real.

(ii) Also, a solution of a differential equation, if exists, is not unique.

For instance, the functions $y = e^{2x}$, $y = 2e^{2x}$, $y = \sqrt{8}e^{2x}$ are solutions of same equation $\frac{dy}{dx} - 2y = 0$. In fact, $y = ce^{2x}$, $c \in \mathbb{R}$, are all solutions of the differential equation $\frac{dy}{dx} - 2y = 0$.

Thus, to represent all possible solutions of a differential equation, we introduce the notion of the general solution of a differential equation.

Definition 10.10 : (General solution)

The solution which contains as many arbitrary constants as the order of the differential equation is called the **general solution**

Remark

The **general solution** includes all possible solutions and typically includes arbitrary constants (in the case of an ODE) or arbitrary functions (in the case of a PDE.)

Definition 10.11 : (Particular solution)

If we give particular values to the arbitrary constants in the general solution of differential equation, the resulting solution is called a Particular Solution.

Remark

(i) Often we find a particular solution to a differential equation by giving extra conditions.

(ii) The general solution of a first order differential equation $y' = f(x, y)$ represents a one-parameter family of curves in xy-plane.

For instance, $y = ce^{2x}$, $c \in \mathbb{R}$, is the general solution of the differential equation $\frac{dy}{dx} - 2y = 0$.

For instance, we have already seen that $y = a\cos x + b\sin x$ satisfies the second order differential equation $\frac{d^2y}{dx^2} + y = 0$. Since it contains two arbitrary constants, it is the general solution of $\frac{d^2y}{dx^2} + y = 0$. If we put $a = 1$, $b = 0$ in the general solution, then we get $y = \cos x$ is a particular solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$.

In application, differential equations do not arise by eliminating the arbitrary constants. They frequently arise while investigating many physical problems in all fields of engineering, science and even in social sciences. Mostly these differential equations are also accompanied by certain conditions on the variables to obtain unique solution satisfying the given conditions.

Example 10.7

Show that $x^2 + y^2 = r^2$, where r is a constant, is a solution of the differential equation $\frac{dy}{dx} = -\frac{x}{y}$.

Solution

Given that $x^2 + y^2 = r^2, r \in \mathbb{R}$ ... (1)

The given equation contains exactly one arbitrary constant.

So, we have to differentiate the given equation once. Differentiate (1) with respect to x, we get

$$2x + 2y\frac{dy}{dx} = 0, \text{ which implies } \frac{dy}{dx} = -\frac{x}{y}.$$

Thus, $x^2 + y^2 = r^2$ satisfies the differential equation $\frac{dy}{dx} = -\frac{x}{y}$.

Hence, $x^2 + y^2 = r^2$ is a solution of the differential equation $\frac{dy}{dx} = -\frac{x}{y}$. ■

Example 10.8

Show that $y = mx + \frac{7}{m}, m \neq 0$ is a solution of the differential equation $xy' + 7\frac{1}{y'} - y = 0$.

Solution

The given function is $y = mx + \frac{7}{m}$, where m is an arbitrary constant. ... (1)

Differentiating both sides of equation (1) with respect to x, we get $y' = m$.

Substituting the values of y' and y in the given differential equation,

we get $xy' + \frac{7}{y'} - y = xm + \frac{7}{m} - mx - \frac{7}{m} = 0$.

Therefore, the given function is a solution of the differential equation $xy' + 7\frac{1}{y'} - y = 0$. ■

Example 10.9

Show that $y = 2\left(x^2 - 1\right) + Ce^{-x^2}$ is a solution of the differential equation $\frac{dy}{dx} + 2xy - 4x^3 = 0$.

Solution

The given function is $y = 2\left(x^2 - 1\right) + Ce^{-x^2}$, where C is an arbitrary constant. ...(1)

Differentiating both sides of equation (1) with respect to x, we get $\frac{dy}{dx} = 4x - 2xCe^{-x^2}$.

Substituting the values of $\frac{dy}{dx}$ and y in the given differential equation, we get

$$\frac{dy}{dx}+2xy-4x^3 = 4x-2xCe^{-x^2}+2x\left[2\left(x^2-1\right)+Ce^{-x^2}\right]-4x^3=0$$

Therefore, the given function is a solution of the differential equation $\frac{dy}{dx}+2xy-4x^3=0$. ■

Example 10.10

Show that $y=a\cos(\log x)+b\sin(\log x), x>0$ is a solution of the differential equation $x^2y''+xy'+y=0$.

Solution

The given function is $y=a\cos(\log x)+b\sin(\log x)$ ...(1)

where a,b are two arbitrary constants. In order to eliminate the two arbitrary constants, we have to differentiate the given function two times successively.

Differentiating equation (1) with respect to x, we get

$$y'=-a\sin(\log x)\cdot\frac{1}{x}+b\cos(\log x)\cdot\frac{1}{x}\Rightarrow xy'=-a\sin(\log x)+b\cos(\log x).$$

Again differentiating this with respect to x, we get

$$xy''+y'=-a\cos(\log x)\cdot\frac{1}{x}-b\sin(\log x)\cdot\frac{1}{x}\Rightarrow x^2y''+xy'+y=0.$$

Therefore, $y=a\cos(\log x)+b\sin(\log x)$ is a solution of the given differential equation. ■

EXERCISE 10.4

1. Show that each of the following expressions is a solution of the corresponding given differential equation.
   (i) $y=2x^2$; $xy'=2y$
   (ii) $y=ae^x+be^{-x}$; $y''-y=0$
2. Find value of m so that the function $y=e^{mx}$ is a solution of the given differential equation.
   (i) $y'+2y=0$ (ii) $y''-5y'+6y=0$
3. The slope of the tangent to the curve at any point is the reciprocal of four times the ordinate at that point. The curve passes through (2,5). Find the equation of the curve.
4. Show that $y=e^{-x}+mx+n$ is a solution of the differential equation $e^x\left(\frac{d^2y}{dx^2}\right)-1=0$.
5. Show that $y=ax+\frac{b}{x}, x\neq 0$ is a solution of the differential equation $x^2y''+xy'-y=0$.
6. Show that $y=ae^{-3x}+b$, where a and b are arbitrary constants, is a solution of the differential equation $\frac{d^2y}{dx^2}+3\frac{dy}{dx}=0$.
7. Show that the differential equation representing the family of curves $y^2=2a\left(x+a^{\frac{2}{3}}\right)$, where a is a positive parameter, is $\left(y^2-2xy\frac{dy}{dx}\right)^3=8\left(y\frac{dy}{dx}\right)^5$.

8. Show that $y = a\cos bx$ is a solution of the differential equation $\frac{d^2y}{dx^2} + b^2y = 0$.

Now, we discuss some standard methods of solving certain type of differential equations of the first order and first degree.

10.6 Solution of First Order and First Degree Differential Equations

10.6.1 Variables Separable Method

In solving differential equations, separation of variables was introduced initially by Leibniz and later it was formulated by John Bernoulli in the year 1694.

A first order differential equation is separable if it can be written as $h(y)y' = g(x)$ where the left side is a product of y' and a function of y and the right side is a function of x. Rewriting a separable differential equation in this form is called the method of separation of variables.

Finding a solution to a first order differential equation will be simple if the variables in the equation can be separated. An equation of the form $f_1(x)g_1(y)dx + f_2(x)g_2(y)dy = 0$ is called an equation with **variable separable** or simply a **separable equation**.

Rewrite the given differential equation as $\frac{f_1(x)}{f_2(x)}dx = -\frac{g_2(y)}{g_1(y)}dy$. ...(1)

Integration of both sides of (1) yields the general solution of the given differential equation as

$\int \frac{f_1(x)}{f_2(x)}dx = -\int \frac{g_2(y)}{g_1(y)}dy + C$, where C is an arbitrary constant.

Remarks

1. No need to add arbitrary constants on both sides as the two arbitrary constants are combined together as a single arbitrary constant.
2. A solution with this arbitrary constant is the general solution of the differential equation.

“Solving a differential equation” is also referred to as “integrating a differential equation”, since the process of finding the solution to a differential equation involves integration.

Example 10.11

Solve $(1+x^2)\frac{dy}{dx} = 1+y^2$.

M5E8B3

Solution

Given that $(1+x^2)\frac{dy}{dx} = 1+y^2$. ... (1)

The given equation is written in the variables separable form

$$\frac{dy}{1+y^2} = \frac{dx}{1+x^2}. \quad \text{... (2)}$$

Integrating both sides of (2), we get $\tan^{-1}y = \tan^{-1}x + C$. ... (3)

But $\tan^{-1}y - \tan^{-1}x = \tan^{-1}\left(\frac{y-x}{1+xy}\right)$. ... (4)

Using (4) in (3) leads to $\tan^{-1}\left(\frac{y-x}{1+xy}\right) = C$, which implies $\frac{y-x}{1+xy} = \tan C = a$ (say).

Thus, $y-x = a(1+xy)$ gives the required solution. ■

Example 10.12

Find the particular solution of $(1+x^3)dy - x^2 y dx = 0$ satisfying the condition $y(1) = 2$.

Solution

$$\text{Given that } (1+x^3)dy - x^2 y dx = 0.$$

$$\text{The above equation is written as } \frac{dy}{y} - \frac{x^2}{1+x^3}dx = 0.$$

$$\text{Integrating both sides gives } \log y - \frac{1}{3}\log(1+x^3) = C_1, \text{ which implies,}$$

$$3\log y - \log(1+x^3) = \log C.$$

$$\text{Thus, } 3\log y = \log(1+x^3) + \log C,$$

$$\text{which reduces to } \log y^3 = \log C(1+x^3).$$

Hence, $y^3 = C(1+x^3)$ gives the general solution of the given differential equation. It is given that when $x=1, y=2$. Then $2^3 = C(1+1) \Rightarrow C = 4$ and hence the particular solution is $y^3 = 4(1+x^3)$. ■

10.6.2 Substitution Method

Let the differential equation be of the form $\frac{dy}{dx} = f(ax+by+c)$.

(i) If $a \neq 0$ and $b \neq 0$, then the substitution $ax+by+c = z$ reduces the given equation to the variables separable form.

(ii) If $a = 0$ or $b = 0$, then the differential equation is already in separable form.

Example 10.13

Solve $y' = \sin^2(x-y+1)$.

Solution

$$\text{Given that } y' = \sin^2(x-y+1)$$

$$\text{Put } z = x-y+1, \text{ so that } \frac{dz}{dx} = 1 - \frac{dy}{dx}.$$

$$\text{Thus, the given equation reduces to } 1 - \frac{dz}{dx} = \sin^2 z.$$

$$\text{i.e., } \frac{dz}{dx} = 1-\sin^2 z = \cos^2 z.$$

$$\text{Separating the variables leads to } \frac{dz}{\cos^2 z} = dx \text{ (or) } \sec^2 z dz = dx.$$

$$\text{On integration, we get } \tan z = x + C \text{ (or) } \tan(x-y+1) = x + C.$$

■

Example 10.14

Solve : $\frac{dy}{dx} = \sqrt{4x+2y-1}$.

Solution

By putting $z = 4x+2y-1$, we have

$$\frac{dz}{dx} = 4+2\frac{dy}{dx} = 4+2\sqrt{z}$$

$$\text{Hence}\quad \frac{dz}{4+2\sqrt{z}} = dx.$$

$$\text{Integrating,}\quad \int\frac{dz}{4+2\sqrt{z}} = x+C.$$

Putting $z = u^2$, we have

$$\int\frac{dz}{4+2\sqrt{z}} = \int\frac{udu}{u+2} = u-2\log|u+2|+C,$$

$$\text{or}\quad \sqrt{z}-2\log\left|\sqrt{z}+2\right| = x+C$$

From which on substituting $z = 4x+2y-1$, we have the general solution

$$\sqrt{4x+2y-1}-2\log\left|\sqrt{4x+2y-1}+2\right| = x+C.$$

Example 10.15

Solve: $\frac{dy}{dx} = \frac{x-y+5}{2(x-y)+7}$.

Solution

Given that $\frac{dy}{dx} = \frac{x-y+5}{2(x-y)+7}$

Put $z = x-y$

$$\frac{dz}{dx} = 1-\frac{dy}{dx}$$

$$\frac{dy}{dx} = 1-\frac{dz}{dx}$$

Thus, the given equation reduces to

$$1-\frac{dz}{dx} = \frac{z+5}{2z+7}$$

$$\frac{dz}{dx} = 1-\frac{z+5}{2z+7}$$

$$\frac{dz}{dx} = \frac{z+2}{2z+7}$$

Separating the variables, we get

$$\frac{2z+7}{z+2}dz = dx$$

$$\frac{2(z+2)+3}{(z+2)}dz = dx$$

$$\left(2+\frac{3}{z+2}\right)dz = dx$$

Integrating both sides, we get

$$2z + 3\log|z+2| = x + C$$

That is, $2(x-y)+3\log|x-y+2| = x+C$.

Example 10.16

Solve : $\frac{dy}{dx} = (3x+y+4)^2$.

Solution

To solve the given differential equation, we make the substitution $3x+y+4 = z$.

Differentiating with respect to x, we get $\frac{dy}{dx} = \frac{dz}{dx} - 3$. So the given differential equation becomes $\frac{dz}{dx} = z^2 + 3$.

In this equation variables are separable. So, separating the variables and integrating, we get the general solution of the given differential equation as $\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{3x+y+4}{\sqrt{3}}\right) = x + C$.

EXERCISE 10.5

1. If F is the constant force generated by the motor of an automobile of mass M, its velocity V is given by $M\frac{dV}{dt} = F - kV$, where k is a constant. Express V in terms of t given that $V = 0$ when $t = 0$.
2. The velocity v, of a parachute falling vertically satisfies the equation $v\frac{dv}{dx} = g\left(1-\frac{v^2}{k^2}\right)$, where g and k are constants. If v and x are both initially zero, find v in terms of x.
3. Find the equation of the curve whose slope is $\frac{y-1}{x^2+x}$ and which passes through the point $(1,0)$.
4. Solve the following differential equations:

(i) $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$ (ii) $ydx + (1+x^2)\tan^{-1}x\, dy = 0$

(iii) $\sin\frac{dy}{dx} = a,\ y(0) = 1$ (iv) $\frac{dy}{dx} = e^{x+y} + x^3e^y$

(v) $\left(e^{y}+1\right)\cos x\,dx+e^{y}\sin x\,dy=0$

(vi) $\left(ydx-xdy\right)\cot\left(\dfrac{x}{y}\right)=ny^{2}dx$

(vii) $\dfrac{dy}{dx}-x\sqrt{25-x^{2}}=0$

(viii) $x\cos y\,dy=e^{x}\left(x\log x+1\right)dx$

(ix) $\tan y\dfrac{dy}{dx}=\cos\left(x+y\right)+\cos\left(x-y\right)$

(x) $\dfrac{dy}{dx}=\tan^{2}\left(x+y\right)$

10.6.3 Homogeneous Form or Homogeneous Differential Equation

Definition 10.12 : (Homogeneous Function of degree n)

A function $f(x,y)$ is said to be a **homogeneous** function of degree n in the variables x and y if, $f(tx,ty)=t^{n}f(x,y)$ for some $n\in\mathbb{R}$ for all suitably restricted x,y and t. This is known as **Euler's homogeneity.**

For instance,

(i) $f(x,y)=6x^{2}+2xy+4y^{2}$ is a homogeneous function in x and y, of degree two.

(ii) But $f(x,y)=x^{3}+\left(\sin x\right)e^{y}$ is not a homogeneous function.

If $f(x,y)$ is a **homogeneous** function of degree zero, then there exists a function g such that $f(x,y)$ is always expressed in the form $g\left(\dfrac{y}{x}\right)$ or $g\left(\dfrac{x}{y}\right)$.

Definition 10.13: (Homogeneous Differential Equation)

An ordinary differential equation is said to be in **homogeneous form**, if the differential equation is written as $\dfrac{dy}{dx}=g\left(\dfrac{y}{x}\right)$.

Caution

The word "homogeneous" used in Definition 10.7 is different from in Definition 10.12.

Remark

(i) The differential equation $M(x,y)dx+N(x,y)dy=0$ [in differential form] is said to be **homogeneous** if M and N are **homogeneous functions of the same degree**.

(ii) The above equation is also written as $\dfrac{dy}{dx}=f(x,y)$ [in derivative form] where $f(x,y)=-M(x,y)/N(x,y)$ is clearly homogeneous of degree 0.

For instance

(1) consider the differential equation $\left(x^{2}-3y^{2}\right)dx+2xy\,dy=0$. The given equation is rewritten as $\dfrac{dy}{dx}=\dfrac{3y^{2}-x^{2}}{2xy}=\dfrac{3}{2}\left(\dfrac{y}{x}\right)-\dfrac{1}{2}\left(\dfrac{1}{y/x}\right)$. Thus, the given equation is expressed as $\dfrac{dy}{dx}=\dfrac{3}{2}\left(\dfrac{y}{x}\right)-\dfrac{1}{2}\left(\dfrac{1}{y/x}\right)=g\left(\dfrac{y}{x}\right)$. Hence, $\left(x^{2}-3y^{2}\right)dx+2xy\,dy=0$ is a homogeneous differential equation.

(2) However, the differential equation $\frac{dy}{dx} = \frac{x^3 + y^2}{2x^3 - xy^2}$ is not homogeneous. (verify!)

To find the solution of a homogeneous differential equation $\frac{dy}{dx} = g\left(\frac{y}{x}\right)$, consider the substitution $v = \frac{y}{x}$. Then, $y = xv$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$. Thus, the given differential equation becomes $x\frac{dv}{dx} = f(v) - v$ which is solved using variables separable method. This leads to the following result.

Theorem 10.1

If $M(x, y)dx + N(x, y)dy = 0$ is a homogeneous equation, then the change of variable $y = vx$, transforms into a separable equation in the variables v and x.

Example 10.17

Solve $\left(x^2 - 3y^2\right)dx + 2xydy = 0$.

Solution

We know that the given equation is homogeneous.

Now, we rewrite the given equation as $\frac{dy}{dx} = \frac{3y}{2x} - \frac{x}{2y}$.

Taking $y = vx$, we have $v + x\frac{dv}{dx} = \frac{3v}{2} - \frac{1}{2v}$ or $x\frac{dv}{dx} = \frac{v^2 - 1}{2v}$.

Separating the variables, we obtain $\frac{2vdv}{v^2 - 1} = \frac{dx}{x}$.

On integration, we get $\log\left|v^2 - 1\right| = \log|x| + \log|C|$,

Hence $\left|v^2 - 1\right| = |Cx|$, where C is an arbitrary constant.

Now, replace v by $\frac{y}{x}$ to get $\left|\frac{y^2}{x^2} - 1\right| = |Cx|$.

Thus, we have $\left|y^2 - x^2\right| = \left|Cx^3\right|$.

Hence, $y^2 - x^2 = \pm Cx^3$ (or) $y^2 - x^2 = kx^3$ gives the general solution.

Example 10.18

Solve $\left(y + \sqrt{x^2 + y^2}\right)dx - xdy = 0, \quad y(1) = 0$.

Solution

The given differential equation is homogeneous (verify!).

Now, we rewrite the given equation in differential form $\frac{dy}{dx} = \frac{y + \sqrt{x^2 + y^2}}{x}$.

Since the initial value of x is 1, we consider $x > 0$ and take $x = \sqrt{x^2}$.

We have $\dfrac{dy}{dx}=\dfrac{y}{x}+\sqrt{1+\left(\dfrac{y}{x}\right)^2}$.

Let $y=vx$. Then, $v+x\dfrac{dv}{dx}=v+\sqrt{1+v^2}$, which becomes $x\dfrac{dv}{dx}=\sqrt{1+v^2}$.

By separating variables, we have $\dfrac{dv}{\sqrt{v^2+1}}=\dfrac{dx}{x}$.

Upon integration, we get $\log\left|v+\sqrt{v^2+1}\right|=\log|x|+\log|C|$ or $v+\sqrt{v^2+1}=xC$.

Now, we replace v by $\dfrac{y}{x}$, we get $\dfrac{y}{x}+\sqrt{\dfrac{y^2}{x^2}+1}=Cx$ (or) $y+\sqrt{x^2+y^2}=Cx^2$ gives the general solution of the given differential equation.

To determine the value of C, we use the condition that $y=0$ when $x=1$. So, we get $C=1$.

Thus $y+\sqrt{x^2+y^2}=x^2$ is the particular solution of the given differential equation. ■

Example 10.19

Solve $(2x+3y)dx+(y-x)dy=0$.

Solution

The given equation can be written as $\dfrac{dy}{dx}=\dfrac{2x+3y}{x-y}$.

This is a homogeneous equation.

Let $y=vx$. Then we have $v+x\dfrac{dv}{dx}=\dfrac{2+3v}{1-v}$.

Thus, $x\dfrac{dv}{dx}=\dfrac{2+2v+v^2}{1-v}$ or $\dfrac{1-v}{(1+v)^2+1}dv=\dfrac{dx}{x}$ or $-\dfrac{1}{2}\left[\dfrac{2v+2}{v^2+2v+2}-\dfrac{4}{(v+1)^2+1}\right]dv=\dfrac{dx}{x}$.

Integrating both sides, we get $-\dfrac{1}{2}\log\left|v^2+2v+2\right|+2\tan^{-1}(v+1)=\log|x|+\log|C|$

or $\log\left|v^2+2v+2\right|-4\tan^{-1}(v+1)=-2\log|x|-2\log|C|$

or $\log\left|v^2+2v+2\right|+\log|x|^2-4\tan^{-1}(v+1)=-2\log|C|$

or $\log\left|(v^2+2v+2)x^2\right|-4\tan^{-1}(v+1)=-2\log|C|$.

Now replacing v by $\dfrac{y}{x}$, we get, $\log\left|y^2+2xy+2x^2\right|-4\tan^{-1}\left(\dfrac{x+y}{x}\right)=k$, where $k=-2\log|C|$ gives the required solution. ■

Example 10.20

Solve $y^2 + x^2 \dfrac{dy}{dx} = xy\dfrac{dy}{dx}$.

Solution

The given equation is rewritten as $\dfrac{dy}{dx} = \dfrac{y^2}{xy - x^2}$.

This is a homogeneous differential equation.

Put $y = vx$. Then, we have $x\dfrac{dv}{dx} = \dfrac{v}{v-1}$.

By separating the variables, $\dfrac{v-1}{v}dv = \dfrac{dx}{x}$.

Integrating, we obtain $v - \log|v| = \log|x| + \log|C|$ or $v = \log|vxC|$.

Replacing v by $\dfrac{y}{x}$, we get, $\dfrac{y}{x} = \log|Cy|$ or $|Cy| = e^{y/x}$ or $y = ke^{y/x}$ (how!) which is the required solution. ∎

Example 10.21

Solve $\left(1 + 2e^{x/y}\right)dx + 2e^{x/y}\left(1 - \dfrac{x}{y}\right)dy = 0$.

Solution

The given equation can be written as $\dfrac{dx}{dy} = \dfrac{\left(\dfrac{x}{y} - 1\right)2e^{x/y}}{1 + 2e^{x/y}} = g\left(\dfrac{x}{y}\right)$. ...(1)

The appearance of $\dfrac{x}{y}$ in equation (1), suggests that the appropriate substitution is $x = vy$.

Put $x = vy$. Then, we have $y\dfrac{dv}{dy} = -\dfrac{2e^v + v}{1 + 2e^v}$.

By separating the variables, we have $\dfrac{1 + 2e^v}{v + 2e^v}dv = -\dfrac{dy}{y}$.

On integration, we obtain

$\log|2e^v + v| = -\log|y| + \log|C|$ or $\log|2ye^v + vy| = \log|C|$ or $2ye^v + vy = \pm C$.

Replace v by $\dfrac{x}{y}$ to get, $2ye^{x/y} + x = k$, where $k = \pm C$, which gives the required solution. ∎

EXERCISE 10.6

Solve the following differential equations:

1. $\left[x + y\cos\left(\dfrac{y}{x}\right)\right]dx = x\cos\left(\dfrac{y}{x}\right)dy$
2. $\left(x^3 + y^3\right)dy - x^2 y\,dx = 0$

3. $ye^{\frac{x}{y}}dx = \left(xe^{\frac{x}{y}} + y\right)dy$

4. $2xydx + \left(x^2 + 2y^2\right)dy = 0$

5. $\left(y^2 - 2xy\right)dx = \left(x^2 - 2xy\right)dy$

6. $x\dfrac{dy}{dx} = y - x\cos^2\left(\dfrac{y}{x}\right)$

7. $\left(1 + 3e^{\frac{y}{x}}\right)dy + 3e^{\frac{y}{x}}\left(1 - \dfrac{y}{x}\right)dx = 0$, given that $y = 0$ when $x = 1$

8. $\left(x^2 + y^2\right)dy = xy\,dx$. It is given that $y(1) = 1$ and $y(x_0) = e$. Find the value of x_0.

10.7 First Order Linear Differential Equations

A **first order differential equation** of the form

$$\frac{dy}{dx} + Py = Q. \qquad \text{... (1)}$$

where P and Q are functions of x only. Here no product of y and its derivative $\frac{dy}{dx}$ occur and the dependent variable y and its derivative with respect to independent variable x occurs only in the first degree.

To integrate (1), let us consider the homogeneous equation $\frac{dy}{dx} + Py = 0$. ...(2)

The equation (2) can be integrated as follows:

Separating the variables, $\dfrac{dy}{y} = -Pdx$.

On integration, we get $ye^{\int Pdx} = C$.

$$\text{Now, } \frac{d}{dx}\left(ye^{\int Pdx}\right) = e^{\int Pdx}\frac{dy}{dx} + y.Pe^{\int Pdx}$$

$$= e^{\int Pdx}\left(\frac{dy}{dx} + Py\right) = Qe^{\int Pdx} \qquad \text{... (3) (using (1))}$$

Integrating both sides of (3) with respect to x, we get the solution of the given differential equation as

$$ye^{\int Pdx} = \int Qe^{\int Pdx}dx + C.$$

Here $e^{\int Pdx}$ is known as the **integrating factor (I.F.)** of (1).

Remarks

1. The solution of linear differential equation is
   $y \times (I.F) = \int Q(I.F)dx + C$, where C is an arbitrary constant.
2. In the integrating factor $e^{\int Pdx}$, P is the coefficient of y in the differential equation provided the coefficient of $\frac{dy}{dx}$ is unity.

E7Y3S4

3. A first order differential equation of the form $\frac{dx}{dy}+Px=Q$, where P and Q are functions of y only. Here no product of x and its derivative $\frac{dx}{dy}$ occur and the dependent variable x and its derivative with respect to independent variable y occurs only in the first degree.

In this case, the solution is given by $xe^{\int Pdy} = \int Qe^{\int Pdy}dy + C$.

Example 10.22

Solve $\frac{dy}{dx}+2y=e^{-x}$.

Solution

$$\text{Given that } \frac{dy}{dx}+2y = e^{-x} \qquad \text{... (1)}$$

This is a linear differential equation.

Here $P=2$; $Q=e^{-x}$.

$$\int Pdx = \int 2dx = 2x.$$

$$\text{Thus, I.F. } = e^{\int Pdx} = e^{2x}.$$

$$\text{Hence the solution of (1) is } ye^{\int Pdx} = \int Qe^{\int Pdx}dx + C.$$

That is, $ye^{2x} = \int e^{-x}e^{2x}dx + C$ or $ye^{2x} = e^{x} + C$ or $y = e^{-x} + Ce^{-2x}$ is the required solution. ■

Example 10.23

Solve $\left[y(1-x\tan x)+x^2\cos x\right]dx - xdy = 0$.

Solution

The given equation can be rewritten as $\frac{dy}{dx}+\frac{(x\tan x-1)}{x}y = x\cos x$.

This is a linear differential equation. Here $P=\frac{(x\tan x-1)}{x}$; $Q=x\cos x$.

$$\int Pdx = \int\frac{(x\tan x-1)}{x}dx = -\log|\cos x| - \log|x| = -\log|x\cos x| = \log\frac{1}{|x\cos x|}.$$

$$\text{Thus, I.F. } = e^{\int Pdx} = e^{\log\frac{1}{|x\cos x|}} = \frac{1}{x\cos x}$$

$$\text{Hence the solution is } ye^{\int Pdx} = \int Qe^{\int Pdx}dx + C$$

$$\text{i.e., } y\frac{1}{x\cos x} = \int (x\cos x)\frac{1}{x\cos x}dx + C$$

or $y\dfrac{1}{x\cos x} = x + C$

or $y = x^2\cos x + Cx\cos x$ is the required solution.

Example 10.24

Solve : $\dfrac{dy}{dx} + 2y\cot x = 3x^2\operatorname{cosec}^2 x$.

Solution

Given that the equation is $\dfrac{dy}{dx} + 2y\cot x = 3x^2\operatorname{cosec}^2 x$.

This is a linear differential equation. Here, $P = 2\cot x$; $Q = 3x^2\operatorname{cosec}^2 x$.

$$\int Pdx = \int 2\cot x dx = 2\log|\sin x| = \log|\sin x|^2 = \log\sin^2 x.$$

$$\text{Thus, I.F} = e^{\int Pdx} = e^{\log\sin^2 x} = \sin^2 x.$$

$$\text{Hence, the solution is. } ye^{\int Pdx} = \int Qe^{\int Pdx}dx + C.$$

$$\text{That is, } y\sin^2 x = \int 3x^2\operatorname{cosec}^2 x\cdot\sin^2 x dx + C = \int 3x^2 dx + C = x^3 + C.$$

Hence, $y\sin^2 x = x^3 + C$ is the required solution.

Example 10.25

Solve $\left(1+x^3\right)\dfrac{dy}{dx} + 6x^2 y = 1 + x^2$.

Solution

Here, to make the coefficient of $\dfrac{dy}{dx}$ unity, divide both sides by $\left(1+x^3\right)$.

Then the equation is $\dfrac{dy}{dx} + \dfrac{6x^2 y}{1+x^3} = \dfrac{1+x^2}{1+x^3}$.

This is a linear differential equation in y.

$$\text{Here, } P = \frac{6x^2}{1+x^3}; Q = \frac{1+x^2}{1+x^3}$$

$$\int Pdx = \int\frac{6x^2}{1+x^3}dx = 2\log\left|1+x^3\right| = \log\left|1+x^3\right|^2 = \log\left(1+x^3\right)^2$$

$$\text{Thus, I.F.} = e^{\int Pdx} = e^{\log\left(1+x^3\right)^2} = \left(1+x^3\right)^2$$

$$\text{Hence the solution is } ye^{\int Pdx} = \int Qe^{\int Pdx}dx + C.$$

$$\text{That is, } y\left(1+x^3\right)^2 = \int\frac{1+x^2}{1+x^3}\left(1+x^3\right)^2 dx + C = \int\left(1+x^2\right)\left(1+x^3\right)dx + C = \int\left(1+x^2+x^3+x^5\right)dx + C$$

or $y\left(1+x^3\right)^2 = x+\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^6}{6}+C$

and $y = \frac{1}{(1+x^3)^2}\left[x+\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^6}{6}+C\right]$ is the required solution. ■

Example 10.26

Solve $ye^y dx = \left(y^3+2xe^y\right)dy$.

Solution

The given equation can be written as $\frac{dx}{dy}-\frac{2}{y}x = y^2e^{-y}$.

This is a linear differential equation. Here $P=-\frac{2}{y};\ Q=y^2e^{-y}$.

$$\int pdy = \int -\frac{2}{y}dy = -2\log|y| = \log|y|^{-2} = \log\left(\frac{1}{y^2}\right),$$

$$\text{Thus, I.F.} = e^{\int Pdy} = e^{\log\left(\frac{1}{y^2}\right)} = \frac{1}{y^2}.$$

$$\text{Hence the solution is } xe^{\int Pdy} = \int Qe^{\int Pdy}dy + C$$

$$\text{That is, } x\left(\frac{1}{y^2}\right) = \int y^2e^{-y}\left(\frac{1}{y^2}\right)dy + C = \int e^{-y}dy + C = -e^{-y}+C$$

or $x = -y^2e^{-y}+Cy^2$ is the required solution. ■

EXERCISE 10.7

Solve the following Linear differential equations:

1. $\cos x\frac{dy}{dx}+y\sin x=1$
2. $\left(1-x^2\right)\frac{dy}{dx}-xy=1$
3. $\frac{dy}{dx}+\frac{y}{x}=\sin x$
4. $\left(x^2+1\right)\frac{dy}{dx}+2xy=\sqrt{x^2+4}$
5. $\left(2x-10y^3\right)dy+ydx=0$
6. $x\sin x\frac{dy}{dx}+\left(x\cos x+\sin x\right)y=\sin x$
7. $\left(y-e^{\sin^{-1}x}\right)\frac{dx}{dy}+\sqrt{1-x^2}=0$
8. $\frac{dy}{dx}+\frac{y}{(1-x)\sqrt{x}}=1-\sqrt{x}$
9. $\left(1+x+xy^2\right)\frac{dy}{dx}+\left(y+y^3\right)=0$
10. $\frac{dy}{dx}+\frac{y}{x\log x}=\frac{\sin 2x}{\log x}$
11. $(x+a)\frac{dy}{dx}-2y=\left(x+a\right)^4$
12. $\frac{dy}{dx}=\frac{\sin^2 x}{1+x^3}-\frac{3x^2}{1+x^3}y$
13. $x\frac{dy}{dx}+y=x\log x$
14. $x\frac{dy}{dx}+2y-x^2\log x=0$
15. $\frac{dy}{dx}+\frac{3y}{x}=\frac{1}{x^2}$, given that $y=2$ when $x=1$

10.8 Applications of First Order Ordinary Differential Equations

The subject of differential equations has vast applications in solving real world problems. The solutions of the differential equations are used to predict the behaviors of the system at a future time, or at an unknown location. In several problems, the rate at which a quantity changes is a given function of the quantity and /or the time. The objective is to find the quantity itself. If x denotes the amount of the quantity present at time t, then the instantaneous rate at which the quantity changes at time t is $\frac{dx}{dt}$. This leads to a differential equation of the form $\frac{dx}{dt} = f(x,t)$. In this section we shall consider this type of problems only. Further, by rate, we mean the instantaneous rate only.

10.8.1 Population growth

Now, we consider the growth of a population (for example, human, an animal, or a bacteria colony) as a function of time t.

Let $x(t)$ be the size of the population at any time t. Although $x(t)$ is integer-valued, we approximate $x(t)$ as a differentiable function and techniques of differential equation can be applied to determine $x(t)$. Assume that population grows at a rate directly proportional to the amount of population present at that time. Then, we obtain

$$\frac{dx}{dt} = kx \text{, where } k \text{ is the constant of proportionality .} \qquad \dots (1)$$

Here $k > 0$, since the population always increases.

The solution of the differential equation is $x(t) = Ce^{kt}$, where C is a constant of integration. The values of C and k are determined with the help of initial conditions. Thus, the population increases exponentially with time. This law of population growth is called **Malthusian law**.

Example 10.27

The growth of a population is proportional to the number present. If the population of a colony doubles in 50 years, in how many years will the population become triple?

Solution

Let $x(t)$ be the population at time t. Then $\frac{dx}{dt} = kx$.

By separating the variables, we obtain $\frac{dx}{x} = kdt$.

Integrating on both sides, we get, $\log|x| = kt + \log|C|$ or $x = Ce^{kt}$, where C is an arbitrary constant.

Let x_0 be the population when $t = 0$ and obtain $C = x_0$.

Thus, we get $x = x_0 e^{kt}$.

Now $x = 2x_0$, when $t = 50$ and thus, $k = \frac{1}{50}\log 2$.

Hence, $x = x_0 2^{\frac{t}{50}}$ is the population at any time t.

Assume that the population is tripled in t_1 years.

That is, $x = 3x_0$, when $t = t_1$.

Thus, $t_1 = 50\left(\frac{\log 3}{\log 2}\right)$. Therefore, the population is tripled in $50\left(\frac{\log 3}{\log 2}\right)$ years. ■

10.8.2. Radioactive decay

The nucleus of an atom consists of combinations of protons and neutrons. Many of these combinations of protons and neutrons are unstable, that is the atoms decay or transmute into the atoms of another substance. Such nuclei are said to be **radioactive**.

It is assumed that the rate $\frac{dA}{dt}$ at which the nuclei of a substance decays is proportional to the amount $A(t)$ of the substance remaining at time t.

Thus, the required differential equation is $\frac{dA}{dt} \propto A$ or $\frac{dA}{dt} = kA$ ...(2), where k is the constant of proportionality. Here $k < 0$, since decay occurs.

Remarks

From equations (1) and (2), we see that the differential equations are the same, but the difference is only in the interpretations of the symbols and the constants of proportionality. For growth as we expect in (1), $k > 0$ and in the case of (2) for decay, $k < 0$.

A single differential equation can serve as a mathematical model for many different phenomena.

Example 10.28

A radioactive isotope has an initial mass 200mg, which two years later is 50mg. Find the expression for the amount of the isotope remaining at any time. What is its half-life? (half-life means the time taken for the radioactivity of a specified isotope to fall to half its original value).

Solution

Let A be the mass of the isotope remaining after t years, and let $-k$ be the constant of proportionality, where $k > 0$. Then the rate of decomposition is modeled by $\frac{dA}{dt} = -kA$, where the minus sign indicates that the mass is decreasing. It is a separable equation. Separating the variables, we get $\frac{dA}{A} = -kdt$

Integrating on both sides, we get $\log|A| = -kt + \log|C|$ or $A = Ce^{-kt}$.

Given that the initial mass is 200mg. That is, $A = 200$ when $t = 0$ and thus, $C = 200$.

Thus, we get $A = 200e^{-kt}$.

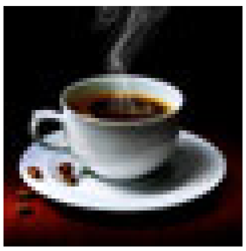
Also, $A = 150$ when $t = 2$ and therefore, $k = \frac{1}{2}\log\left(\frac{4}{3}\right)$.

Hence, $A(t) = 200e^{-\frac{t}{2}\log\left(\frac{4}{3}\right)}$ is the mass of isotope remaining after t years.

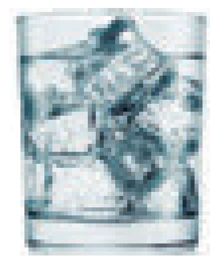
The half-life t_h is the time corresponding to $\mathrm{A} = 100\mathrm{mg}$.

Thus, $t_h = \dfrac{2\log\left(\frac{1}{2}\right)}{\log\left(\frac{3}{4}\right)}$.

10.8.3. Newton's Law of cooling/warming



Consider pouring a $150°$ cup of coffee and kept it on the table in an $80°C$ room. What happens to the temperature of the coffee? We observe that the cup of coffee will cool off until it reaches the room temperature.



Now consider taking a $35°$ glass of cold water from the refrigerator and kept it on the table in an $80°C$ room. What happens to the temperature of the cold water? Similarly, we can observe the water will warm up until it reaches room temperature.

According to **Newton's law of cooling or warming**, the rate at which the temperature of a body changes is proportional to the difference between the temperature of the body and the temperature of the surrounding medium the so-called **ambient temperature**. If $T(t)$ represents the temperature of a body at time t, T_m the temperature of the surrounding medium, and $\frac{dT}{dt}$ the rate at which the temperature of the body changes, then Newton's law of cooling(or warming) is $\frac{dT}{dt} \propto T - T_m$ or $\frac{dT}{dt} = k\left(T - T_m\right)$, where k is constant of proportionality.

In either case, cooling or warming, if T_m is constant, it stands to reason that $k < 0$.

Example 10.29

In a murder investigation, a corpse was found by a detective at exactly 8 p.m. Being alert, the detective also measured the body temperature and found it to be 70°F. Two hours later, the detective measured the body temperature again and found it to be 60°F. If the room temperature is 50°F, and assuming that the body temperature of the person before death was 98.6°F, at what time did the murder occur?

$\left[\log(2.43) = 0.88789; \quad \log(0.5) = -0.69315\right]$

Solution

Let T be the temperature of the body at any time t and with time 0 taken to be 8 p.m.

By Newton's law of cooling, $\frac{dT}{dt} = k(T - 50)$ or $\frac{dT}{T - 50} = k\,dt$.

Integrating on both sides, we get $\log|50 - T| = kt + \log C$ or $50 - T = Ce^{kt}$.

When $t = 0$, $T = 70$, and so $C = -20$

When $t = 2, T = 60$, we have $-10 = -20e^{k2}$.

Thus, $k = \frac{1}{2}\log\left(\frac{1}{2}\right)$.

Hence, the solution is $50 - T = -20e^{\frac{1}{2}t\log\left(\frac{1}{2}\right)}$ or $T = 50 + 20\left(\frac{1}{2}\right)^{\frac{t}{2}}$

Now, we would like to find the value of t, for which $T(t) = 98.6$, and $t = 2\left(\dfrac{\log\left(\frac{48.6}{20}\right)}{\log\left(\frac{1}{2}\right)}\right) \approx -2.56$

It appears that the person was murdered at about 5.30 p.m. ■

10.8.4 Mixture problems

Mixing problems occur quite frequently in chemical industry. Now we explain how to solve the basic model involving a single tank.

A substance S is allowed to flow into a certain mixture in a container at a constant rate, and the mixture is kept uniform by stirring. Further, in one such situation, this uniform mixture simultaneously flows out of the container at another rate. Now we seek to determine the quantity of the substance S present in the mixture at time t.

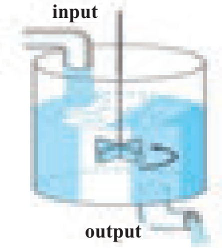


Fig. 10.2

Letting x to denote the amount of S present at time t and the derivative $\frac{dx}{dt}$ to denote the rate of change of x with respect to t. If IN denotes the rate at which S enters the mixture and OUT denotes the rate at which it leaves, then we have the equation $\frac{dx}{dt} = IN - OUT$

Example 10.30

A tank contains 1000 litres of water in which 100 grams of salt is dissolved. Brine (*Brine is a high-concentration solution of salt (usually sodium chloride) in water*) runs in a rate of 10 litres per minute, and each litre contains 5grams of dissolved salt. The mixture of the tank is kept uniform by stirring. Brine runs out at 10 litres per minute. Find the amount of salt at any time t.

Solution

Let $x(t)$ denote the amount of salt in the tank at time t. Its rate of change is $\frac{dx}{dt} = \text{in flow rate} - \text{out flow rate}$

Now, 5 grams times 10 litres gives an inflow of 50 grams of salt. Also, the out flow of brine is 10 litres per minute. This is $10/1000 = 0.01$ of the total brine content in the tank. Hence, the outflow of salt is 0.01 times $x(t)$, that is $0.01x(t)$.

Thus the differential equation for the model is $\frac{dx}{dt} = 50 - 0.01x = -0.01(x - 5000)$

This can be written as $\frac{dx}{x - 5000} = -(0.01)dt$

Integrating both sides, we obtain $\log|x - 5000| = -0.01t + \log C$

or $x - 5000 = Ce^{-0.01t}$ or $x = 5000 + Ce^{-0.01t}$

Initially, when $t = 0$, $x = 100$, so $100 = 5000 + C$. Thus, $C = -4900$.

Hence, the amount of the salt in the tank at time t is $x = 5000 - 4900e^{-0.01t}$.

EXERCISE 10.8

1. The rate of increase in the number of bacteria in a certain bacteria culture is proportional to the number present. Given that the number triples in 5 hours, find how many bacteria will be present after 10 hours?
2. Find the population of a city at any time t, given that the rate of increase of population is proportional to the population at that instant and that in a period of 40 years the population increased from 3,00,000 to 4,00,000.
3. The equation of electromotive force for an electric circuit containing resistance and self-inductance is $E = Ri + L\dfrac{di}{dt}$, where E is the electromotive force is given to the circuit, R the resistance and L, the coefficient of induction. Find the current i at time t when $E = 0$.
4. The engine of a motor boat moving at 10 m/s is shut off. Given that the retardation at any subsequent time (after shutting off the engine) equal to the velocity at that time. Find the velocity after 2 seconds of switching off the engine.
5. Suppose a person deposits ₹10,000 in a bank account at the rate of 5% per annum compounded continuously. How much money will be in his bank account 18 months later?
6. Assume that the rate at which radioactive nuclei decay is proportional to the number of such nuclei that are present in a given sample. In a certain sample 10% of the original number of radioactive nuclei have undergone disintegration in a period of 100 years. What percentage of the original radioactive nuclei will remain after 1000 years?
7. Water at temperature $100^\circ C$ cools in 10 minutes to $80^\circ C$ in a room temperature of $25^\circ C$. Find
   (i) The temperature of water after 20 minutes
   (ii) The time when the temperature is $40^\circ C$
   $$\left[\log_e \frac{11}{15} = -0.3101;\ \log_e 5 = 1.6094\right]$$
8. At 10.00 A.M. a woman took a cup of hot instant coffee from her microwave oven and placed it on a nearby Kitchen counter to cool. At this instant the temperature of the coffee was $180^\circ F$, and 10 minutes later it was $160^\circ F$. Assume that constant temperature of the kitchen was $70^\circ F$.
   (i) What was the temperature of the coffee at 10.15A.M.? $\left[\log \dfrac{9}{11} = -0.6061\right]$
   (ii) The woman likes to drink coffee when its temperature is between $130^\circ F$ and $140^\circ F$. between what times should she have drunk the coffee? $\left[\log \dfrac{6}{11} = -0.2006\right]$
9. A pot of boiling water at $100^\circ C$ is removed from a stove at time $t = 0$ and left to cool in the kitchen. After 5 minutes, the water temperature has decreased to $80^\circ C$, and another 5 minutes later it has dropped to $65^\circ C$. Determine the temperature of the kitchen.

10. A tank initially contains 50 litres of pure water. Starting at time $t = 0$ a brine containing with 2 grams of dissolved salt per litre flows into the tank at the rate of 3 litres per minute. The mixture is kept uniform by stirring and the well-stirred mixture simultaneously flows out of the tank at the same rate. Find the amount of salt present in the tank at any time $t > 0$.

EXERCISE 10.9

B3K7M7

Choose the correct or the most suitable answer from the given four alternatives :

1. The order and degree of the differential equation $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^{1/3} + x^{1/4} = 0$ are respectively

   (1) 2, 3 (2) 3, 3 (3) 2, 6 (4) 2, 4

2. The differential equation representing the family of curves $y = A\cos(x + B)$, where A and B are parameters, is

   (1) $\frac{d^2y}{dx^2} - y = 0$ (2) $\frac{d^2y}{dx^2} + y = 0$ (3) $\frac{d^2y}{dx^2} = 0$ (4) $\frac{d^2x}{dy^2} = 0$

3. The order and degree of the differential equation $\sqrt{\sin x}\,(dx + dy) = \sqrt{\cos x}\,(dx - dy)$ is

   (1) 1, 2 (2) 2, 2 (3) 1, 1 (4) 2, 1

4. The order of the differential equation of all circles with centre at (h, k) and radius 'a' is

   (1) 2 (2) 3 (3) 4 (4) 1

5. The differential equation of the family of curves $y = Ae^{x} + Be^{-x}$, where A and B are arbitrary constants is

   (1) $\frac{d^2y}{dx^2} + y = 0$ (2) $\frac{d^2y}{dx^2} - y = 0$ (3) $\frac{dy}{dx} + y = 0$ (4) $\frac{dy}{dx} - y = 0$

6. The general solution of the differential equation $\frac{dy}{dx} = \frac{y}{x}$ is

   (1) $xy = k$ (2) $y = k\log x$ (3) $y = kx$ (4) $\log y = kx$

7. The solution of the differential equation $2x\frac{dy}{dx} - y = 3$ represents

   (1) straight lines (2) circles (3) parabola (4) ellipse

8. The solution of $\frac{dy}{dx} + p(x)y = 0$ is

   (1) $y = ce^{\int p dx}$ (2) $y = ce^{-\int p dx}$ (3) $x = ce^{-\int p dy}$ (4) $x = ce^{\int p dy}$

9. The integrating factor of the differential equation $\frac{dy}{dx} + y = \frac{1 + y}{\lambda}$ is

   (1) $\frac{x}{e^{\lambda}}$ (2) $\frac{e^{\lambda}}{x}$ (3) λe^{x} (4) e^{x}

10. The integrating factor of the differential equation $\frac{dy}{dx}+P(x)y=Q(x)$ is x, then $P(x)$

(1) x (2) $\frac{x^2}{2}$ (3) $\frac{1}{x}$ (4) $\frac{1}{x^2}$

11. The degree of the differential equation $y(x)=1+\frac{dy}{dx}+\frac{1}{1\cdot 2}\left(\frac{dy}{dx}\right)^2+\frac{1}{1\cdot 2\cdot 3}\left(\frac{dy}{dx}\right)^3+\ldots.$ is

(1) 2 (2) 3 (3) 1 (4) 4

12. If p and q are the order and degree of the differential equation $y\frac{dy}{dx}+x^3\left(\frac{d^2y}{dx^2}\right)+xy=\cos x$, when

(1) $p<q$ (2) $p=q$ (3) $p>q$ (4) p exists and q does not exist

13. The solution of the differential equation $\frac{dy}{dx}+\frac{1}{\sqrt{1-x^2}}=0$ is

(1) $y+\sin^{-1}x=c$ (2) $x+\sin^{-1}y=0$ (3) $y^2+2\sin^{-1}x=C$ (4) $x^2+2\sin^{-1}y=0$

14. The solution of the differential equation $\frac{dy}{dx}=2xy$ is

(1) $y=Ce^{x^2}$ (2) $y=2x^2+C$ (3) $y=Ce^{-x^2}+C$ (4) $y=x^2+C$

15. The general solution of the differential equation $\log\left(\frac{dy}{dx}\right)=x+y$ is

(1) $e^x+e^y=C$ (2) $e^x+e^{-y}=C$ (3) $e^{-x}+e^y=C$ (4) $e^{-x}+e^{-y}=C$

16. The solution of $\frac{dy}{dx}=2^{y-x}$ is

(1) $2^x+2^y=C$ (2) $2^x-2^y=C$ (3) $\frac{1}{2^x}-\frac{1}{2^y}=C$ (4) $x+y=C$

17. The solution of the differential equation $\frac{dy}{dx}=\frac{y}{x}+\frac{\phi\left(\frac{y}{x}\right)}{\phi'\left(\frac{y}{x}\right)}$ is

(1) $x\phi\left(\frac{y}{x}\right)=k$ (2) $\phi\left(\frac{y}{x}\right)=kx$ (3) $y\phi\left(\frac{y}{x}\right)=k$ (4) $\phi\left(\frac{y}{x}\right)=ky$

18. If $\sin x$ is the integrating factor of the linear differential equation $\frac{dy}{dx}+Py=Q$, then P is

(1) $\log\sin x$ (2) $\cos x$ (3) $\tan x$ (4) $\cot x$

19. The number of arbitrary constants in the general solutions of order n and $n+1$ are respectively

(1) $n-1,\ n$ (2) $n,\ n+1$ (3) $n+1,\ n+2$ (4) $n+1,\ n$

20. The number of arbitrary constants in the particular solution of a differential equation of third order is

(1) 3 (2) 2 (3) 1 (4) 0

21. Integrating factor of the differential equation $\frac{dy}{dx}=\frac{x+y+1}{x+1}$ is

(1) $\frac{1}{x+1}$ (2) $x+1$ (3) $\frac{1}{\sqrt{x+1}}$ (4) $\sqrt{x+1}$

22. The population P in any year t is such that the rate of increase in the population is proportional to the population. Then

(1) $P=Ce^{kt}$ (2) $P=Ce^{-kt}$ (3) $P=Ckt$ (4) $P=C$

23. P is the amount of certain substance left in after time t. If the rate of evaporation of the substance is proportional to the amount remaining, then

(1) $P=Ce^{kt}$ (2) $P=Ce^{-kt}$ (3) $P=Ckt$ (4) $Pt=C$

24. If the solution of the differential equation $\frac{dy}{dx}=\frac{ax+3}{2y+f}$ represents a circle, then the value of a is

(1) 2 (2) -2 (3) 1 (4) -1

25. The slope at any point of a curve $y=f(x)$ is given by $\frac{dy}{dx}=3x^2$ and it passes through (-1,1). Then the equation of the curve is

(1) $y=x^3+2$ (2) $y=3x^2+4$ (3) $y=3x^3+4$ (4) $y=x^3+5$

SUMMARY

1. A differential equation is any equation which contains at least one derivative of an unknown function, either ordinary derivative or partial derivative.
2. The **order** of a differential equation is the highest derivative present in the differential equation.
3. If a differential equation is expressible in a polynomial form, then the integral power of the highest order derivative appears is called the **degree** of the differential equation
4. If a differential equation is not expressible to polynomial equation form having the highest order derivative as the leading term then that the degree of the differential equation is not defined.
5. If a differential equation contains only ordinary derivatives of one or more functions with respect to a single independent variable, it is said to be an ordinary differential equation (ODE).
6. An equation involving only partial derivatives of one or more functions of two or more independent variables is called a partial differential equation (PDE).
7. The result of eliminating one arbitrary constant yields a first order differential equation and that of eliminating two arbitrary constants leads to a second order differential equation and so on.
8. A solution of a differential equation is an expression for the dependent variable in terms of the independent variable(s) which satisfies the differential equation.
9. The solution which contains as many arbitrary constants as the order of the differential equation is called the **general solution**
10. If we give particular values to the arbitrary constants in the general solution of differential equation, the resulting solution is called a Particular Solution.

11. An equation of the form $f_1(x)g_1(y)dx + f_2(x)g_2(y)dy = 0$ is called an equation with variable separable or simply a separable equation.
12. A function $f(x, y)$ is said to be a **homogeneous** function of degree n in the variables x and y if, $f(tx, ty) = t^n f(x, y)$ for some $n \in \mathbb{R}$ for all suitably restricted x, y and t. This is known as **Euler's homogeneity.**
13. If $f(x, y)$ is a homogeneous function of degree zero, then there exists a function g such that $f(x, y)$ is always expressed in the form $g\left(\frac{y}{x}\right)$.
14. An ordinary differential equation is said to be in homogeneous form, if the differential equation is written as $\frac{dy}{dx} = g\left(\frac{y}{x}\right)$.
15. The differential equation $M(x, y)dx + N(x, y)dy = 0$ [in differential form] is said to be **homogeneous** if M and N are **homogeneous functions of the same degree**.
16. A **first order differential equation** of the form $\frac{dy}{dx} + Py = Q$.
    where P and Q are functions of x only. Here no product of y and its derivative $\frac{dy}{dx}$ occurs and the dependent variable y and its derivative with respect to independent variable x occur only in the first degree.
    The solution of the given differential equation (1) is given by $ye^{\int P dx} = \int Qe^{\int P dx} dx + C$.
    Here $e^{\int P dx}$ is known as the integrating factor (I.F.)
17. A first order differential equation of the form $\frac{dx}{dy} + Px = Q$, where P and Q are functions of y only. Here no product of x and its derivative $\frac{dx}{dy}$ occurs and the dependent variable x and its derivative with respect to independent variable y occur only in the first degree.In this case, the solution is given by $xe^{\int P dy} = \int Qe^{\int P dy} dy + C$.
18. If x denotes the amount of the quantity present at time t, then the instantaneous rate at which the quantity changes at time t is $\frac{dx}{dt}$.
    This leads to a differential equation of the form $\frac{dx}{dt} = f(x, t)$.

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B262_12_MATHS_EM

Chapter 11

Probability Distributions

Probability theory is nothing but common sense reduced to calculation

-Laplace

I1W4Z3

Laplace
(1749-1827)

The history of random variables and how they evolved into mapping from sample space to real numbers was a subject of interest. The modern interpretation certainly occurred after the invention of sets and maps (1900), but as Eremenko says, random variables were used much earlier. Mathematicians felt the need to interpret random variables as maps. In 1812, Laplace published his book on *Theory analytique des probabilities* in which he laid down many fundamental results in statistics. The first half of this treatise was concerned with probability methods and problems and the second half with statistical applications.

Learning Objectives

Upon completion of this chapter, students will be able to

- define a random variable, discrete and continuous random variables
- define probability mass (density) function
- determine probability mass (density) function from cumulative distribution function
- obtain cumulative distribution function from probability mass (density) function
- calculate mean and variance for random variable
- identify and apply Bernoulli and binomial distributions.

11.1 Introduction

The concept of a sample space that completely describes the possible outcomes of a random experiment has been developed in volume 2 of I year higher secondary course.

In this chapter, we learn about a function, called random variable defined on the sample space of a random experiment and its probability distribution.

11.2 Random Variable

The outcome from a random experiment is not always a simple thing to represent in notion. In many random experiments that we have considered, the sample space S has been a description of possible outcomes. That is the outcome of an experiment, or the points in the sample space S, need not be numbers. For example in the random experiment of tossing a coin, the outcomes are H (head) or T (tail). It is necessary to deal with numerical values, in some situation, for outcomes of random experiment. Therefore, we assign a number to each outcome of the experiment say 1to head and 0 to tail. Such an assignment of numerical values to the elements in S is called a *random variable*. *A random variable is a function*. Thus, a random variable is:

Definition 11.1

A random variable *X is a function defined on a sample space S into the real numbers $\mathbb{R}$ such that the inverse image of points or subset or interval of $\mathbb{R}$ is an event in S, for which probability is assigned.*

We use the capital letters of the alphabet, such as X, Y, and Z to represent the random variables and the small letters, such as x, y, and z to represent the possible values of the random variables.

Suppose $S = \{\omega_1, \omega_2, \omega_3, \ldots\}$ is the sample space of a random experiment and $\mathbb{R}$ denotes the real line. Then the random variable X is a real valued function defined on S and is denoted by $X : S \to \mathbb{R}$. If ω is a sample point in S, then $X(\omega)$ is a real number.

The range set is the collection of $X(\omega)$ such that $\omega \in S$. That is the range set denoted by R_x is $R_x = \{X(\omega) / \omega \in S\}$.

Fig 11.1 shows the mapping of some sample points ω_i or events of the *Sample space* S on the *real line R*.

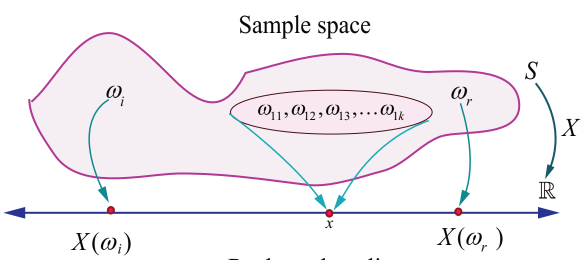


Fig. 11.1

For instance, if x is a possible value of X for $\omega_{11}, \omega_{12}, \omega_{13}, \ldots \omega_{1k} \in S$, then $\{\omega_{11}, \omega_{12}, \omega_{13}, \ldots \omega_{1k}\}$ is called inverse image of x.

That is $X^{-1}(x) = \{\omega_{11}, \omega_{12}, \omega_{13}, \ldots \omega_{1k}\}$ is an event in S

Illustration 11.1

Suppose a coin is tossed once. The sample space consists of two sample points H (head) and T (tail).

That is $S = \{T, H\}$

Let $X : S \to \mathbb{R}$ be the number of heads

Then $X(T) = 0$, and $X(H) = 1$.

Thus X is a random variable that takes on the values 0 and 1. If $X(\omega)$ denotes the number of heads, then

$$X(\omega) = \begin{cases} 0 & \text{for } \omega = \text{Tail} \\ 1 & \text{for } \omega = \text{Head} \end{cases}$$

Example 11.1

Suppose two coins are tossed once. If X denotes the number of tails, (i) write down the sample space (ii) find the inverse image of 1 (iii) the values of the random variable and number of elements in its inverse images.

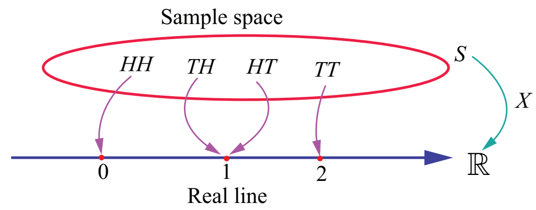
Solution

(i) The sample space $S = \{H, T\} \times \{H, T\}$

That is $S = \{TT, TH, HT, HH\}$

(ii) Let $X : S \to \mathbb{R}$ be the number of tails

$$\begin{aligned}
\text{Then } X(TT) &= 2 && \text{(2 Tails)}\\
X(TH) &= 1 && \text{(1 Tail)}\\
X(HT) &= 1 && \text{(1 Tail)}\\
\text{and } X(HH) &= 0 && \text{(0 Tails).}
\end{aligned}$$



A mapping $X(.)$ from S to $\mathbb{R}$

Fig. 11.2

Then X is a random variable that takes on the values 0, 1 and 2.

Let $X(\omega)$ denote the number of tails, this gives

$$X(\omega) = \begin{cases} 2 & \text{if } \omega = TT \\ 1 & \text{if } \omega = HT, TH \\ 0 & \text{if } \omega = HH \end{cases}$$

The inverse images of 1 is $\{TH, HT\}$. That is $X^{-1}(\{1\}) = \{TH, HT\}$.

(iii) Number of elements in inverse images are shown in the table.

Values of the Random Variable	0	1	2	Total
Number of elements in inverse image	1	2	1	4

Example 11.2

Suppose a pair of unbiased dice is rolled once. If X denotes the total score of two dice, write down (i) the sample space (ii) the values taken by the random variable X, (iii) the inverse image of 10, and (iv) the number of elements in inverse image of X.

Solution

(i) The sample space

S = $\{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\}$,

consists of 36 ordered pairs (α, β) where α and β can take any integer value between 1 and 6 as shown. X is assigned to each point (α, β) the sum of the numbers on the dice .

$$S = \left\{\begin{array}{llllll}
(1, 1), & (1, 2), & (1, 3), & (1, 4), & (1, 5), & (1, 6) \\
(2, 1), & (2, 2), & (2, 3), & (2, 4), & (2, 5), & (2, 6) \\
(3, 1), & (3, 2), & (3, 3), & (3, 4), & (3, 5), & (3, 6) \\
(4, 1), & (4, 2), & (4, 3), & (4, 4), & (4, 5), & (4, 6) \\
(5, 1), & (5, 2), & (5, 3), & (5, 4), & (5, 5), & (5, 6) \\
(6, 1), & (6, 2), & (6, 3), & (6, 4), & (6, 5), & (6, 6)
\end{array}\right\}$$

That is $X(\alpha, \beta) = \alpha + \beta$.

Therefore

$$\begin{aligned}
X(1,1) &= 1+1 = 2\\
X(1,2) &= X(2,1) = 3\\
X(1,3) &= X(2,2) = X(3,1) = 4\\
X(1,4) &= X(2,3) = X(3,2) = X(4,1) = 5\\
X(1,5) &= X(2,4) = X(3,3) = X(4,2) = X(5,1) = 6\\
X(1,6) &= X(2,5) = X(3,4) = X(4,3) = X(5,2) = X(6,1) = 7
\end{aligned}$$

$$X(2,6) = X(3,5)=X(4,4)=X(5,3)=X(6,2)=8$$
$$X(3,6) = X(4,5)=X(5,4)=X(6,3)=9$$
$$X(4,6) = X(5,5)=X(6,4)=10$$
$$X(5,6) = (6,5)=11$$
$$X(6,6) = 12.$$

(ii) Then the random variable X takes on the values 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.

(iii) The inverse images of 10 is $\{(4,\ 6),(5,\ 5),(6,\ 4)\}$.

(iv) The number of inverse images are given below

Values of the random variable	2	3	4	5	6	7	8	9	10	11	12	Total
Number of elements in inverse image	1	2	3	4	5	6	5	4	3	2	1	36

Example 11.3

An urn contains 2 white balls and 3 red balls. A sample of 3 balls are chosen at random from the urn. If X denotes the number of red balls chosen, find the values taken by the random variable X and its number of inverse images.

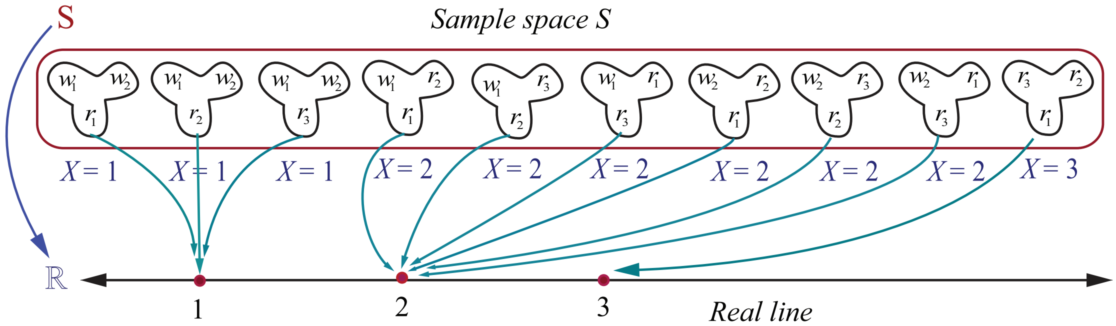
Solution

Let us denote white and red balls as $w_1, w_2, r_1, r_2,$ and r_3.

The sample space consists of $5c_3 = 10$ different samples of size 3.

That is $S=\{w_1w_2r_1,\ w_1w_2r_2,\ w_1w_2r_3,\ w_1r_1r_2,\ w_1r_2r_3,\ w_1r_1r_3,\ w_2r_1r_2,\ w_2r_2r_3,\ w_2r_1r_3,\ r_1r_2r_3\}$.

The random variable X takes on the values 1, 2, and 3.



A mapping X (.) from S to real numbers

Fig. 11.3

Values of the Random Variable X	1	2	3	Total
Number of elements in inverse images	3	6	1	10

Remark

If X denotes the number of white balls, then X takes on the values 0,1, and 2 and the elements in inverse images are

Values of the Random Variable X	0	1	2	Total
Number of elements in inverse images	1	6	3	10

Illustration 11.2

A batch of 150 students is taken in 4 buses to an excursion. There are 38 students in the first bus, 36 in second bus, 32 in the third bus, and the remaining students in the fourth bus. When the buses arrive at the destination, one of the 150 students is randomly chosen.

Suppose that X denotes the number of students on the bus of that randomly chosen student. Then X takes on the values 32, 36, 38, and 44.

Example 11. 4

Two balls are chosen randomly from an urn containing 6 white and 4 black balls. Suppose that we win ₹ 30 for each black ball selected and we lose ₹ 20 for each white ball selected. If X denotes the winning amount, find the values of X and number of points in its inverse images.

Solution

The possible events of selection are (i) both balls may be black, or (ii) one white and one black or (iii) both are white. Therefore X is a random variable that take the values,

$$X \text{ (both are black balls)} = ₹\ 2(30) = ₹\ 60$$

$$X \text{ (one black and one white ball)} = ₹\ 30 - ₹\ 20 = ₹\ 10$$

$$X \text{ (both are white balls)} = ₹\ 2(-20) = -₹\ 40$$

Therefore X takes on the values 60,10, and $-$ 40.

Note : The inverse image of 40 is $\{b_1b_2,\ b_1b_3,\ b_1b_4,\ b_2b_3,\ b_2b_4,\ b_3b_4.\}$

Values of the Random Variable X	60	10	– 40	Total
Number of elements in inverse images	6	24	15	45
	$\binom{6}{0}\binom{4}{2}$	$\binom{6}{1}\binom{4}{1}$	$\binom{6}{2}\binom{4}{0}$	$\binom{10}{2}$

Illustration 11.3

A coin is tossed until head occurs.

The sample space is $S = \{H, TH, TTH, TTTH, \ldots\}$.

Suppose X denotes the number of times the coin is tossed until head occurs.

Then the random variable X takes on the values $1, 2, 3, \ldots$

Illustration 11.4

Suppose N is the number of customers in the queue that arrive at a service desk during a time period, then the sample space should be the set of non-negative integers. That is $S = \{0, 1, 2, 3, \ldots\}$ and N is a random variable that takes on the values $0, 1, 2, 3, \ldots$

Illustration 11.5

If an experiment consists in observing the lifetime of an electrical bulb, then a sample space would be the life time of electrical bulb. Therefore the sample space is $S = [0, \infty)$. Suppose X denotes the lifetime of the bulb, then X is a random variable that takes on the values in $[0, \infty)$.

Illustration 11.6

Let D be a disk of radius r. Suppose a point is chosen at random in D. Let X denote the distance of the point from the centre. Then the sample space $S = D$ and X is the random variable that takes on any number from 0 to r. That is $X(\omega) \in [0, r]$, for $\omega \in S$.

EXERCISE 11.1

1. Suppose X is the number of tails occurred when three fair coins are tossed once simultaneously. Find the values of the random variable X and number of points in its inverse images.
2. In a pack of 52 playing cards, two cards are drawn at random simultaneously. If the number of black cards drawn is a random variable, find the values of the random variable and number of points in its inverse images.
3. An urn contains 5 mangoes and 4 apples. Three fruits are taken at random. If the number of apples taken is a random variable, then find the values of the random variable and number of points in its inverse images.
4. Two balls are chosen randomly from an urn containing 6 red and 8 black balls. Suppose that we win ₹ 15 for each red ball selected and we lose ₹ 10 for each black ball selected. If X denotes the winning amount, find the values of X and number of points in its inverse images.
5. A six sided die is marked '2' on one face, '3' on two of its faces, and '4' on remaining three faces. The die is thrown twice. If X denotes the total score in two throws, find the values of the random variable and number of points in its inverse images.

11.3 Types of Random Variable

In this chapter we shall restrict our study to two types of random variables, one is a random variable assuming at most a countable number of values and another is a random variable assuming the values continuously. That is

(i) Discrete Random variable (for counting the quantity)

(ii) Continuous Random variable (for measuring the quantity)

11.3.1 Discrete random variables

In this section we discuss

(i) Discrete random variables

(ii) Probability mass function

(iii) Cumulative distribution function.

(iv) Obtaining cumulative distribution function from probability mass function.

(v) Obtaining probability mass function from cumulative distribution function.

If the range set of the random variables is discrete set of numbers then the inverse image of random variable is either finite or countably infinite. Such a random variable is called discrete random variable. A random variable defined on a discrete sample space is discrete.

Definition 11.2 (Discrete Random Variable)

A random variable X is defined on a sample space S into the real numbers $\mathbb{R}$ is called discrete random variable if the range of X is countable, that is, it can assume only a finite or countably infinite number of values, where every value in the set S has positive probability with total one.

Remark

It is also possible to define a discrete random variable on continuous sample space. For instance,

(i) for a continuous sample space $S = [0,1]$, the random variable defined by $X(\omega) = 10$ for all $\omega \in S$, is a discrete random variable.

(ii) for a continuous sample space $S = [0, 20]$, the random variable defined by

$$X(\omega) = \begin{cases} 1 & \text{for } \omega \in [0,10) \\ 2 & \text{for } \omega \in [10,20] \end{cases}$$ is a discrete random variable.

11.3.2 Probability Mass Function

The probability that a discrete random variable X takes on a particular value x, that is $P(X = x)$, is frequently denoted by $f(x)$ **or** $p(x)$. The function $f(x)$ is typically called the probability mass function, although some authors also refer to it as the probability function or the frequency function. In this chapter, when the random variable is discrete, the common terminology the probability mass function is used and its common abbreviation is pmf.

Definition 11.3 (Probability mass function)

If X is a discrete random variable with discrete values $x_1, x_2, x_3, \ldots x_n \ldots,$ then the function denoted by $f(.)$ or $p(.)$ and defined by

$$f(x_k) = P(X = x_k), \quad \text{for } k = 1, 2, 3, \ldots n, \ldots$$

is called the probability mass function of X

Theorem 11.1 (Without proof)

The function $f(x)$ is a probability mass function if and only if it satisfies the following properties for the set of real values $x_1, x_2, x_3, \ldots x_n \ldots$.

(i) $f(x_k) \geq 0$ for $k = 1, 2, 3, \ldots n, \ldots$ and (ii) $\sum_k f(x_k) = 1$

Note:

(i) The set of probabilities $\{f(x_k) = P(X = x_k), \quad k = 1, 2, 3, \ldots n, \ldots\}$ is also known as **probability distribution of discrete random variable**

(ii) Since the random variable is a function, it can be presented

(a) in tabular form (b) in graphical form and (c) in an expression form

Example 11.5

Two fair coins are tossed simultaneously (equivalent to a fair coin is tossed twice). Find the probability mass function for number of heads occurred.

Solution

The sample space $S = \{H, T\} \times \{H, T\}$

That is $S = \{TT, TH, HT, HH\}$

Let X be the random variable denoting the number of heads.

Therefore

$$X(TT) = 0, \qquad X(TH) = 1,$$
$$X(HT) = 1, \text{ and } \quad X(HH) = 2.$$

Then the random variable X takes on the values 0, 1 and 2

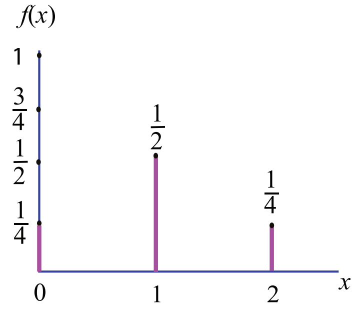
Values of the Random Variable	0	1	2	Total
Number of elements in inverse images	1	2	1	4

The probabilities are given by

$$f(0) = P(X=0) = \frac{1}{4},$$

$$f(1) = P(X=1) = \frac{1}{2}$$

$$\text{and } f(2) = P(X=2) = \frac{1}{4}$$



Probability mass function of $f(x)$

Fig. 11.4

The function $f(x)$ satisfies the conditions

(i) $f(x) \geq 0$, for $x = 0, 1, 2$

(ii) $\sum_x f(x) = \sum_{x=0}^{x=2} f(x) = f(0) + f(1) + f(2)$

$$= \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1$$

Therefore $f(x)$ is a probability mass function.

The probability mass function is given by

x	0	1	2
$f(x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

(or)

$$f(x) = \begin{cases} \frac{1}{4} & \text{for } x = 0 \\ \frac{1}{2} & \text{for } x = 1 \\ \frac{1}{4} & \text{for } x = 2 \end{cases}$$

Example 11.6

A pair of fair dice is rolled once. Find the probability mass function to get the number of fours.

Solution

Let X be a random variable whose values x are the number of fours.

The sample space S is given in the table.

$$S = \left\{ \begin{matrix} (1,1), & (1,2), & (1,3), & (1,4), & (1,5), & (1,6) \\ (2,1), & (2,2), & (2,3), & (2,4), & (2,5), & (2,6) \\ (3,1), & (3,2), & (3,3), & (3,4), & (3,5), & (3,6) \\ (4,1), & (4,2), & (4,3), & (4,4), & (4,5), & (4,6) \\ (5,1), & (5,2), & (5,3), & (5,4), & (5,5), & (5,6) \\ (6,1), & (6,2), & (6,3), & (6,4), & (6,5), & (6,6) \end{matrix} \right\}$$

It can also be written as

$S = \{(i, j)\}$, where $i = 1, 2, 3 \ldots, 6$ and $j = 1, 2, 3, \ldots 6$

Therefore X takes on the values of 0, 1, and 2.

We observe that

(i) $X = 0$, if (i, j) for $i \neq 4$, $j \neq 4$,

(ii) $X = 1$, if $(1,4), (2,4), (3,4), (5,4), (6,4), (4,1), (4,2), (4,3), (4,5), (4,6)$

(iii) $X = 2$, if $(4,4)$,

Therefore,

Values of the Random Variable X	0	1	2	Total
Number of elements in inverse images	25	10	1	36

The probabilities are

$$f(0) = P(X=0) = \frac{25}{36},$$

$$f(1) = P(X=1) = \frac{10}{36}$$

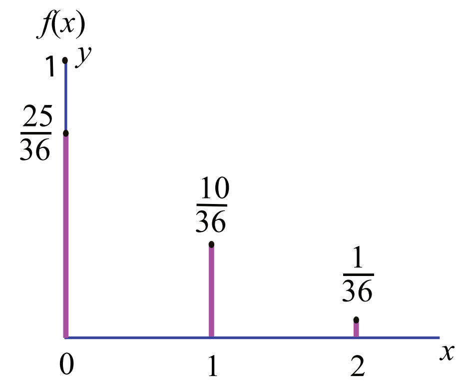
and $$f(2) = P(X=2) = \frac{1}{36}$$

Clearly the function $f(x)$ satisfies the conditions

(i) $f(x) \geq 0$, for $x = 0, 1, 2$ and

(ii) $\sum_{x} f(x) = \sum_{x=0}^{x=2} f(x) = f(0)+f(1)+f(2) = 1$

$$= \frac{25}{36}+\frac{10}{36}+\frac{1}{36} = 1$$



Probability mass function of $f(x)$
Fig. 11.5

The probability mass function is presented as

x	0	1	2
$f(x)$	$\frac{25}{36}$	$\frac{10}{36}$	$\frac{1}{36}$

(or) $$f(x) = \begin{cases} \frac{25}{36} & \text{for } x = 0 \\ \frac{10}{36} & \text{for } x = 1 \\ \frac{1}{36} & \text{for } x = 2 \end{cases}$$

11.3.3 Cumulative Distribution Function or Distribution Function

There are many situations to compute the probability that the observed value of a random variable X will be less than or equal to some real number x. Writing $F(x) = P(X \leq x)$ for every real number x, we call $F(x)$, the cumulative distribution function or distribution function of the random variable X and its common abbreviation is cdf.

Definition 11.4: (cumulative distribution function)

The **cumulative distribution function** $F(x)$ of a discrete random variable X, taking the values $x_1, x_2, x_3, \ldots$ such that $x_1 < x_2 < x_3 < \cdots$ with probability mass function $f(x_i)$ is

$$F(x) = P(X \leq x) = \sum_{x_i \leq x} f(x_i), \qquad x \in \mathbb{R}$$

The distribution function of a discrete random variable is known as Discrete Distribution Function. Although, the probability mass function $f(x)$ is defined *only* for a set of discrete values $x_1, x_2, x_3, \ldots$, the cumulative distribution function $F(x)$ is defined for all real values of $x \in \mathbb{R}$.

We can compute the cumulative distribution function using the probability mass function

$$F(x) = P(X \leq x) = \sum_{x_i \leq x} f(x_i) = \sum_{x_i \leq x} P(X = x_i)$$

If X takes only a finite number of values $x_1, x_2, x_3, \ldots x_n$, where $x_1 < x_2 < x_3 <, \cdots < x_n$, then the cumulative distribution function is given by

$$F(x) = \begin{cases} 0, & -\infty < x < x_1 \\ f(x_1), & x_1 \le x < x_2 \\ f(x_1)+f(x_2), & x_2 \le x < x_3 \\ f(x_1)+f(x_2)+f(x_3), & x_3 \le x < x_4 \\ \vdots & \vdots \\ f(x_1)+f(x_2)+\ldots+f(x_n), & x_n \le x < \infty \end{cases}$$

For a discrete random variable X, the cumulative distribution function satisfies the following properties.

(i) $0 \le F(x) \le 1$, for all $x \in \mathbb{R}$.

(ii) $F(x)$ is real valued non-decreasing function $(x < y$, then $F(x) \le F(y))$.

(iii) $F(x)$ is right continuous function $\left(\lim_{x \to a+} F(x) = F(a)\right)$.

(iv) $\lim_{x \to -\infty} F(x) = F(-\infty) = 0$.

(v) $\lim_{x \to +\infty} F(x) = F(+\infty) = 1$.

(vi) $P(x_1 < X \le x_2) = F(x_2) - F(x_1)$.

(vii) $P(X > x) = 1 - P(X \le x) = 1 - F(x)$.

(viii) $P(X = x_k) = F(x_k) - F(x_k^-)$.

Note

Some authors use left continuity in the definition of a cumulative distribution function $F(x)$, instead of right continuity.

11.3.4 Cumulative Distribution Function from Probability Mass function

Both the probability mass function and the cumulative distribution function of a discrete random variable X contain all the probabilistic information of X. The *probability distribution* of X is determined by either of them. In fact, the distribution function F of a discrete random variable X can be expressed in terms of the probability mass function $f(x)$ of X and vice versa.

Example 11.7

If the probability mass function $f(x)$ of a random variable X is

x	1	2	3	4
$f(x)$	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{5}{12}$	$\frac{1}{12}$

find (i) its cumulative distribution function, hence find (ii) $P(X \le 3)$ and, (iii) $P(X \ge 2)$

Solution

(i) By definition the cumulative distribution function for discrete random variable is

$$F(x) = P(X \le x) = \sum_{x_i \le x} P(X = x_i)$$

$P(X<1) = 0$ for $-\infty < x < 1$.

$$F(1) = P(X \le 1) = \sum_{x_i \le x} P(X = x_i) = \sum_{-\infty}^{1} P(X = x) = P(X<1) + P(X=1) = 0 + \frac{1}{12} = \frac{1}{12}.$$

$$F(2) = P(X \le 2) = \sum_{-\infty}^{2} P(X = x) = P(X<1) + P(X=1) + P(X=2).$$

$$= 0 + \frac{1}{12} + \frac{5}{12} = \frac{1}{2}.$$

$$F(3) = P(X \le 3) = \sum_{-\infty}^{3} P(X = x) = P(X<1) + P(X=1) + P(X=2) + P(X=3).$$

$$= 0 + \frac{1}{12} + \frac{5}{12} + \frac{5}{12} = \frac{11}{12}.$$

$$F(4) = P\left(X \le 4\right) = \sum_{-\infty}^{4} P(X = x) = P(X<1) + P(X=1) + P(X=2) + P(X=3) + P(X=4).$$

$$= 0 + \frac{1}{12} + \frac{5}{12} + \frac{5}{12} + \frac{1}{12} = 1.$$

Therefore the cumulative distribution function is

$$\mathrm{F}(x) = \begin{cases} 0, & -\infty < x < 1 \\ \frac{1}{12}, & 1 \le x < 2 \\ \frac{1}{2}, & 2 \le x < 3 \\ \frac{11}{12}, & 3 \le x < 4 \\ 1, & 4 \le x < \infty \end{cases}$$

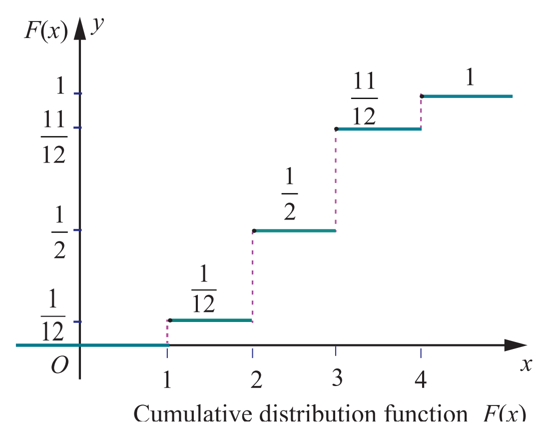


Fig. 11.6

(ii) $P(X \le 3) = F(3) = \frac{11}{12}$.

(iii) $P(X \ge 2) = 1 - P(X<2) = 1 - P(X \le 1) = 1 - F(1) = 1 - \frac{1}{12} = \frac{11}{12}$.

Example 11.8

A six sided die is marked '1' on one face, '2' on two of its faces, and '3' on remaining three faces. The die is rolled twice. If X denotes the total score in two throws.

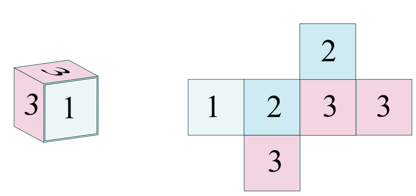


Fig. 11.7

(i) Find the probability mass function.

(ii) Find the cumulative distribution function.

(iii) Find $P(3 \le X < 6)$ (iv) Find $P(X \ge 4)$.

Solution:

Since X denotes the total score in two throws, it takes on the values 2, 3, 4, 5, and 6. From the Sample space **S**, we have

Values of the Random Variable	2	3	4	5	6	Total
Number of elements in inverse images	1	4	10	12	9	36

Sample space S

I \ II	1	2	2	3	3	3
1	2	3	3	4	4	4
2	3	4	4	5	5	5
2	3	4	4	5	5	5
3	4	5	5	6	6	6
3	4	5	5	6	6	6
3	4	5	5	6	6	6

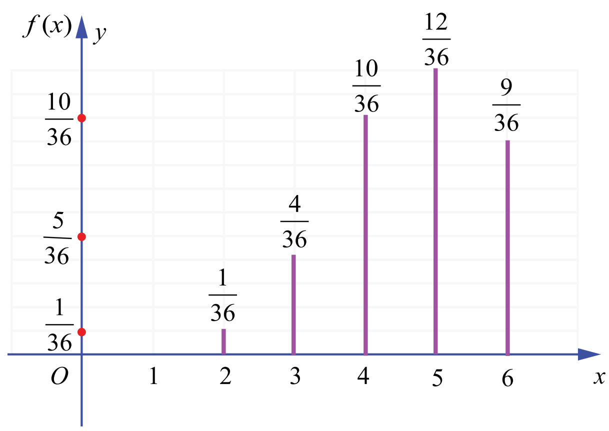
$$P(X=2) = \frac{1}{36}, \qquad P(X=3) = \frac{4}{36}$$

$$P(X=4) = \frac{10}{36}, \qquad P(X=5) = \frac{12}{36}, \text{ and}$$

$$P(X=6) = \frac{9}{36}.$$

(i) Probability mass function is

x	2	3	4	5	6
$f(x)$	$\frac{1}{36}$	$\frac{4}{36}$	$\frac{10}{36}$	$\frac{12}{36}$	$\frac{9}{36}$



Probability mass function

Fig. 11.8

(ii) Cumulative distribution function

By definition of the cumulative distribution function for discrete random variable we have

$$F(x) = P(X \le x) = \sum_{x_i \le x} P(X = x_i),$$

$$P(X < x) = 0 \quad \text{for} \quad -\infty < X < 2.$$

$$F(2) = P(X \le 2) = \sum_{-\infty}^{2} P(X = x) = P(X < 2) + P(X = 2) = 0 + \frac{1}{36} = \frac{1}{36}.$$

$$F(3) = P(X \le 3) = \sum_{-\infty}^{3} P(X = x) = P(X < 2) + P(X = 2) + P(X = 3) = 0 + \frac{1}{36} + \frac{4}{36} = \frac{5}{36}.$$

$$F(4) = P(X \le 4) = \sum_{-\infty}^{4} P(X = x) = P(X < 2) + P(X = 2) + P(X = 3) + P(X = 4)$$

$$= 0 + \frac{1}{36} + \frac{4}{36} + \frac{10}{36} = \frac{15}{36}.$$

$$\mathrm{F}(5) = P(X \le 5) = \sum_{-\infty}^{5} P(X = x) = P(X < 2) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5)$$

$$= 0 + \frac{1}{36} + \frac{4}{36} + \frac{10}{36} + \frac{12}{36} = \frac{27}{36}.$$

$$F(6) = P(X \le 6) = \sum_{-\infty}^{6} P(X = x)$$

$$= P(X < 2) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) + P(X = 6)$$

$$= 0 + \frac{1}{36} + \frac{4}{36} + \frac{10}{36} + \frac{12}{36} + \frac{9}{36} = 1.$$

Therefore the cumulative distribution function is

$$F(x) = \begin{cases} 0 & \text{for} \quad -\infty < x < 2 \\ \frac{1}{36} & \text{for} \quad 2 \le x < 3 \\ \frac{5}{36} & \text{for} \quad 3 \le x < 4 \\ \frac{15}{36} & \text{for} \quad 4 \le x < 5 \\ \frac{27}{36} & \text{for} \quad 5 \le x < 6 \\ 1 & \text{for} \quad 6 \le x < \infty \end{cases}$$

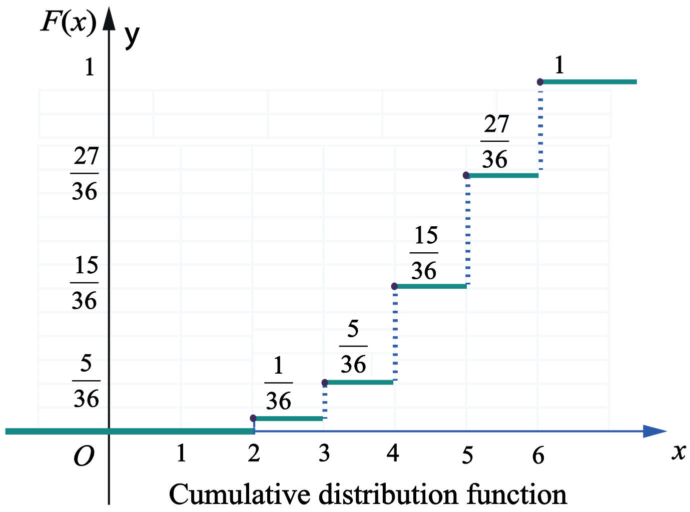


Fig. 11.9

(iii) $P(3 \le X < 6) = \sum_{x=3}^{5} P(X = x_i) = P(X = 3) + P(X = 4) + P(X = 5)$

$$= \frac{4}{36} + \frac{10}{36} + \frac{12}{36} = \frac{26}{36}.$$

(iv) $P(X \ge 4) = \sum_{x=4}^{\infty} P(X = x_i)$

$$= P(X = 4) + P(X = 5) + P(X = 6)$$

$$= \frac{10}{36} + \frac{12}{36} + \frac{9}{36} = \frac{31}{36}.$$

11.3.5 Probability Mass Function from Cumulative Distribution Function

For a discrete random variable X, the cumulative distribution function F has jumps at each of the x_i, and is constant between successive $x_i's$. The height of the jump at x_i is $f(x_i)$; in this way the probability at x_i can be retrieved from F.

Suppose X is a discrete random variable taking the values $x_1, x_2, x_3, \ldots$ such that $x_1 < x_2, < x_3 \cdots$ and $F(x_i)$ is the distribution function. Then the probability mass function $f(x_i)$ is given by

$$f(x_i) = F(x_i) - F(x_{i-1}), \quad i = 1, 2, 3, \ldots$$

Note

The jump of a function $F(x)$ at $x = a$ is $\left|F(a^+) - F(a^-)\right|$. Since F is non-decreasing and continuous to the right, the jump of a cumulative distribution function F is $P(X = x) = F(x) - F(x^-)$.

Here the jump (because of discontinuity) acts as a probability. That is, the set of discontinuities of a cumulative distribution function is at most countable!

Example 11.9

Find the probability mass function $f(x)$ of the discrete random variable X whose cumulative distribution function $F(x)$ is given by

$$F(x) = \begin{cases} 0 & -\infty < x < -2 \\ 0.25 & -2 \le x < -1 \\ 0.60 & -1 \le x < 0 \\ 0.90 & 0 \le x < 1 \\ 1 & 1 \le x < \infty \end{cases}$$

Also find (i) $P(X < 0)$ and (ii) $P(X \ge -1)$.

Solution

Since X is a discrete random variable, from the given data, X takes on the values

$-2, -1, 0,$ and 1.

For discrete random variable X, by definition, we have $f(x) = P(X = x)$

Therefore left hand limit of $F(x)$ at $x = -2$ is $F(-2^-)$

$$f(-2) = P(X = -2) = F(-2) - F(-2^-) = 0.25 - 0 = 0.25.$$

Similarly for other jump points, we have

$$f(-1) = P(X = -1) = F(-1) - F(-2) = 0.60 - 0.25 = 0.35.$$

$$f(0) = P(X = 0) = F(0) - F(-1) = 0.90 - 0.60 = 0.30,$$

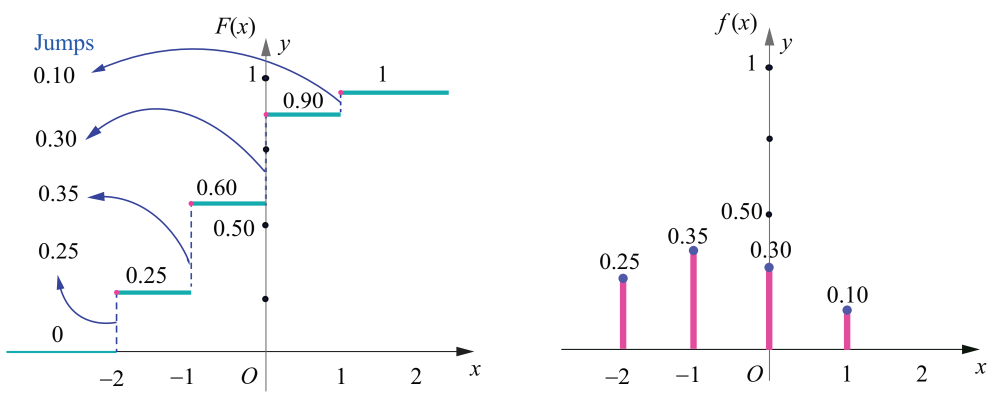
$$f(1) = P(X = 1) = F(1) - F(0) = 1 - 0.90 = 0.10.$$

Therefore the probability mass function is

x	-2	-1	0	1
$f(x)$	0.25	0.35	0.30	0.10

The distribution function $F(x)$ has jumps at $x = -2, -1, 0,$ and 1. The jumps are respectively $0.25, 0.35, 0.30,$ and 0.1 is shown in the figure given below.

These jumps determine the probability mass function



Distribution function $F(x)$ and jumps at each of x_i Probability mass function $f(x)$

Fig. 11.10

(i) $P(X<0) = \sum_{-\infty}^{-1} P(X=x) = P(X=-2)+P(X=-1) = 0.25+0.35 = 0.60$.

(ii) $P(X \geq -1) = \sum_{-1}^{1} P(X=x) = P(X=-1)+P(X=0) + P(X=1) = 0.35+0.30+0.10 = 0.75$ ■

Example 11.10

A random variable X has the following probability mass function.

x	1	2	3	4	5	6
$f(x)$	k	$2k$	$6k$	$5k$	$6k$	$10k$

Find (i) $P(2<X<6)$ (ii) $P(2 \leq X < 5)$ (iii) $P(X \leq 4)$ (iv) $P(3<X)$

Solution

Since the given function is a probability mass function, the total probability is one. That is $\sum_x f(x)=1$.

From the given data $k+2k+6k+5k+6k+10k=1$

$$30k=1 \Rightarrow k=\frac{1}{30}$$

Therefore the probability mass function is

x	1	2	3	4	5	6
$f(x)$	$\frac{1}{30}$	$\frac{2}{30}$	$\frac{6}{30}$	$\frac{5}{30}$	$\frac{6}{30}$	$\frac{10}{30}$

(i) $P(2<X<6) = f(3)+f(4)+f(5) = \frac{6}{30}+\frac{5}{30}+\frac{6}{30} = \frac{17}{30}$.

(ii) $P(2 \leq X < 5) = f(2)+f(3)+f(4) = \frac{2}{30}+\frac{6}{30}+\frac{5}{30} = \frac{13}{30}$.

(iii) $P(X \leq 4) = f(1)+f(2)+f(3)+f(4) = \frac{1}{30}+\frac{2}{30}+\frac{6}{30}+\frac{5}{30} = \frac{14}{30}$.

(iv) $P(3<X) = f(4)+f(5)+f(6) = \frac{5}{30}+\frac{6}{30}+\frac{10}{30} = \frac{21}{30}$. ■

EXERCISE 11.2

1. Three fair coins are tossed simultaneously. Find the probability mass function for number of heads occurred.
2. A six sided die is marked '1' on one face, '3' on two of its faces, and '5' on remaining three faces. The die is thrown twice. If X denotes the total score in two throws, find
   (i) the probability mass function (ii) the cumulative distribution function
   (iii) $P(4 \le X < 10)$ (iv) $P(X \ge 6)$
3. Find the probability mass function and cumulative distribution function of number of girl child in families with 4 children, assuming equal probabilities for boys and girls.
4. Suppose a discrete random variable can only take the values 0, 1, and 2.
   The probability mass function is defined by
   $$f(x) = \begin{cases} \dfrac{x^2+1}{k}, & \text{for } x = 0,1,2 \\ 0 & \text{otherwise} \end{cases}$$
   Find (i) the value of k (ii) cumulative distribution function (iii) $P(X \ge 1)$.
5. The cumulative distribution function of a discrete random variable is given by
   $$F(x) = \begin{cases} 0 & -\infty < x < -1 \\ 0.15 & -1 \le x < 0 \\ 0.35 & 0 \le x < 1 \\ 0.60 & 1 \le x < 2 \\ 0.85 & 2 \le x < 3 \\ 1 & 3 \le x < \infty \end{cases}$$
   Find (i) the probability mass function (ii) $P(X < 1)$ and (iii) $P(X \ge 2)$.
6. A random variable X has the following probability mass function.

x	1	2	3	4	5
$f(x)$	k^2	$2k^2$	$3k^2$	$2k$	$3k$

Find (i) the value of k (ii) $P(2 \le X < 5)$ (iii) $P(3 < X)$

7. The cumulative distribution function of a discrete random variable is given by
   $$F(x) = \begin{cases} 0 & \text{for } -\infty < x < 0 \\ \dfrac{1}{2} & \text{for } 0 \le x < 1 \\ \dfrac{3}{5} & \text{for } 1 \le x < 2 \\ \dfrac{4}{5} & \text{for } 2 \le x < 3 \\ \dfrac{9}{10} & \text{for } 3 \le x < 4 \\ 1 & \text{for } 4 \le x < \infty \end{cases}$$
   Find (i) the probability mass function (ii) $P(X < 3)$ and (iii) $P(X \ge 2)$.

11.4 Continuous Distributions

In this section we learn

(i) Continuous random variable

(ii) Probability density function

(iii) Distribution function (Cumulative distribution function).

(iv) To determine distribution function from probability density function.

(v) To determine probability density function from distribution function.

Sometimes a measurement such as current in a copper wire or length of lifetime of an electric bulb, can assume any value in an interval of real numbers. Then any precision in the measurement is possible. The random variable that represents this measurement is said to be a **continuous** random variable. The range of the random variable includes all values in an interval of real numbers; that is, the range can be thought of as a continuum of real numbers

11.4.1 The definition of continuous random variable

Definition 11.5 (Continuous Random Variable)

*Let S be a sample space and let a random variable $X : S \to R$ that takes on any value in a set I of R. Then X is called a **continuous random variable** if $P\left(X = x\right) = 0$ for every x in I*

11.4.2 Probability density function

Definition 11.6: (Probability density function)

*A non-negative real valued function $f(x)$ is said to be a **probability density function** if, for each possible outcome x, $x \in \left[a, b\right]$ of a continuous random variable X having the property*

$$P(a \le X \le b) = \int_a^b f(x)\ dx$$

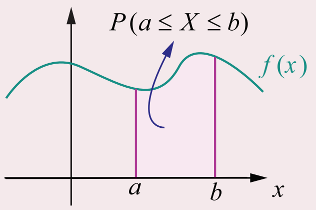


Fig. 11.11

Theorem 11.2 (Without proof)

A function $f(.)$ is a probability density function for some continuous random variable X if and only if it satisfies the following properties.

(i) $f(x) \ge 0$, for every x and (ii) $\int_{-\infty}^{\infty} f(x)dx = 1$.

Note

It follows from the above definition, if X is a continuous random variable,

$$P(a \le X \le b) = \int_a^b f(x)\ dx, \text{which means that } P(X = a) = \int_a^a f(x)\ dx = 0$$

That is probability when X takes on any one particular value is zero.

11.4.3 Distribution function (Cumulative distribution function)

Definition 11.7 : (Cumulative Distribution Function)

*The **distribution function** or **cumulative distribution function** $F(x)$ of a continuous random variable X with probability density $f(x)$ is*

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(u)\,du, \qquad -\infty < u < \infty .$$

Remark

(1) In the discrete case, $f(a) = P(X = a)$ is the probability that X takes the value a.

In the continuous case, $f(x)$ at $x = a$ is not the probability that X takes the value a, that is $f(a) \ne P(X = a)$. If X is continuous type, $P(X = a) = 0$ for $a \in \mathbb{R}$.

(2) When the random variable is continuous, the summation used in discrete is replaced by integration.

(3) For continuous random variable

$P(a < X < b) = P(a \le X < b) = P(a < X \le b) = P(a \le X \le b)$

(4) The distribution function of a continuous random variable is known as Continuous Distribution Function.

11.4.3.1 Properties of distribution function

For a continuous random variable X, the cumulative distribution function satisfies the following properties.

(i) $0 \le F(x) \le 1$.

(ii) $F(x)$ is a real valued non-decreasing. That is, if $x < y$, then $F(x) \le F(y)$.

(iii) $F(x)$ is continuous everywhere.

(iv) $\lim\limits_{x \to -\infty} F(x) = F(-\infty) = 0$ and $\lim\limits_{x \to \infty} F(x) = F(+\infty) = 1$.

(v) $P(X > x) = 1 - P(X \le x) = 1 - F(x)$.

(vi) $P(a < X < b) = F(b) - F(a)$.

Example 11.11

Find the constant C such that the function $f(x) = \begin{cases} Cx^2 & 1 < x < 4 \\ 0 & \text{Otherwise} \end{cases}$

is a density function, and compute (i) $P(1.5 < X < 3.5)$ (ii) $P(X \le 2)$ (iii) $P(3 < X)$.

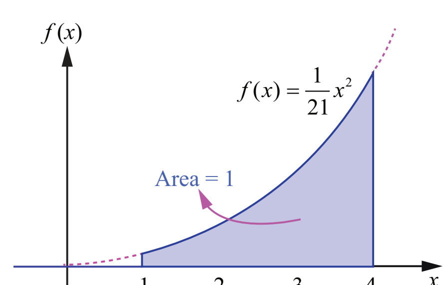
Solution

Since the given function is a probability density function,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$

That is $$\int_{-\infty}^{1} f(x)\,dx + \int_{1}^{4} f(x)\,dx + \int_{4}^{\infty} f(x)\,dx = 1.$$

From the given information,



probability mass function $f(x)$

Fig. 11.12

$$\int_{-\infty}^{1} 0\,dx + \int_{1}^{4} Cx^2\,dx + \int_{4}^{\infty} 0\,dx = 1.$$

$$0 + C\left[\frac{x^3}{3}\right]_1^4 + 0 = 1, \quad \Rightarrow C\left[\frac{64-1}{3}\right] = 1 \quad \Rightarrow 21C = 1 \quad \Rightarrow C = \frac{1}{21}.$$

Therefore the probability density function is

$$f(x) = \begin{cases} \frac{1}{21}x^2 & 1 < x < 4 \\ 0 & \text{Otherwise} \end{cases}$$

Since $f(x)$ is continuous, the probability that X is equal to any particular value is zero. Therefore when the random variable is continuous, either or both of the signs $<$ by $\le$ and $>$ by $\ge$ can be interchanged. Thus

(i) $P(1.5 < X < 3.5) = P(1.5 \le X < 3.5) = P(1.5 < X \le 3.5) = P(1.5 \le X \le 3.5)$

Therefore

$$P(1.5 < X < 3.5) = \int_{1.5}^{3.5} f(x)\,dx = \frac{1}{21}\int_{1.5}^{3.5} x^2\,dx$$

$$= \frac{1}{21}\left(\frac{x^3}{3}\right) = \frac{1}{21}\left(\frac{(3.5)^3 - (1.5)^3}{3}\right)$$

$$= \frac{79}{126}.$$

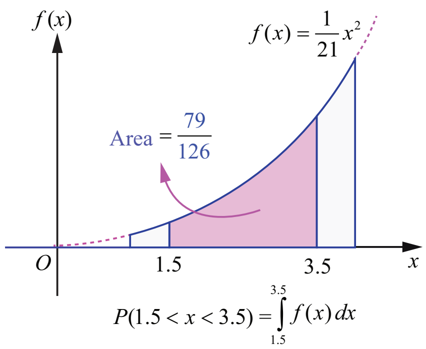


Fig. 11.13

(ii) $P(X \le 2) = \int_{-\infty}^{2} f(x)\,dx = \int_{-\infty}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx$

Therefore

$$P(X \le 2) := 0 + \frac{1}{21}\int_{1}^{2} x^2\,dx = \frac{1}{21}\left(\frac{x^3}{3}\right)_1^2$$

$$= \frac{1}{21}\left(\frac{2^3 - 1^3}{3}\right) = \frac{7}{63}.$$

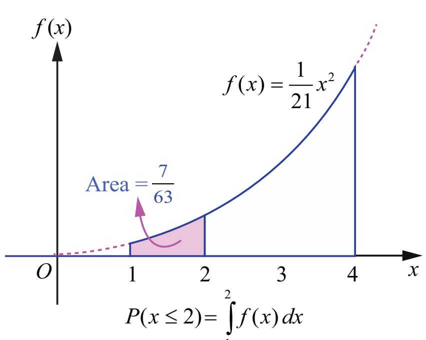


Fig. 11.14

(iii) $P(3 < X) = \int_{3}^{\infty} f(x)\,dx = \int_{3}^{4} f(x)\,dx + \int_{4}^{\infty} f(x)\,dx$

$$= \frac{1}{21}\int_{3}^{4} x^2\,dx + 0 = \frac{1}{21}\left(\frac{x^3}{3}\right)_3^4.$$

$$= \frac{1}{21}\left(\frac{4^3 - 3^3}{3}\right) = \frac{37}{63}$$

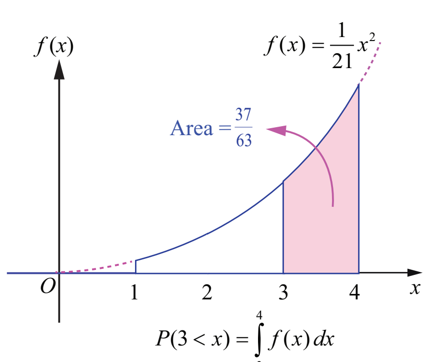


Fig. 11.15

11.4.4 Distribution function from Probability density function

Both the probability density function and the cumulative distribution function (or distribution function) of a continuous random variable X contain all the probabilistic information of X. The *probability distribution* of X is determined by either of them. Let us learn the method to determine the distribution function F of a continuous random variable X from the probability density function $f(x)$ of X and vice versa.

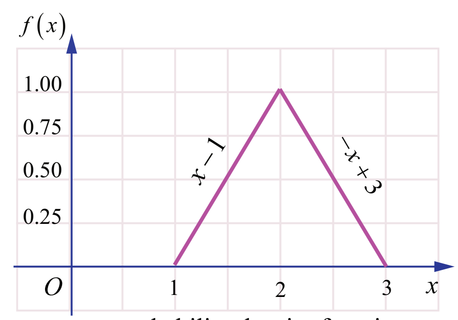
Example 11.12

If X is the random variable with probability density function $f(x)$ given by,

$$f(x)=\begin{cases} x-1, & 1\le x<2 \\ -x+3, & 2\le x<3 \\ 0 & \text{otherwise} \end{cases}$$

find (i) the distribution function $F(x)$

(ii) $P(1.5\le X\le 2.5)$



probability density function

Fig. 11.16

Solution

(i) By definition $F(x)=P(X\le x)=\int_{-\infty}^{x} f(u)\,du$

K4P7U6

When $x<1$ $\quad F(x) = P(X\le x)=\int_{-\infty}^{x} 0\,du=0$.

When $1\le x<2$ $\quad F(x) = P(X\le x)=\int_{-\infty}^{1} 0\,du+\int_{1}^{x}(u-1)\,du$

$$=0+\left[\frac{(u-1)^2}{2}\right]_1^x=\frac{(x-1)^2}{2}$$

When $2\le x<3$ $\quad F(x) = P(X\le x)=\int_{-\infty}^{1} 0\,du+\int_{1}^{2}(u-1)\,du+\int_{2}^{x}(3-u)\,du$

$$=0+\left[\frac{(u-1)^2}{2}\right]_1^2+\left[-\frac{(3-u)^2}{2}\right]_2^x$$

$$=\frac{1^2-0}{2}+\frac{1-(3-x)^2}{2}=1-\frac{(3-x)^2}{2}$$

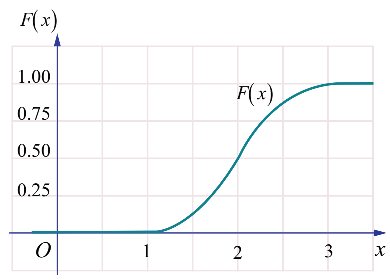
When $x\ge 3$, $\quad F(x) = P(X\le x)=\int_{-\infty}^{1} 0\,du+\int_{1}^{2}(u-1)\,du+\int_{2}^{3}(3-u)\,du+\int_{3}^{x} 0\,du$

$$=\int_{-\infty}^{1} 0\,du+\int_{1}^{2}(u-1)\,du+\int_{2}^{3}(3-u)\,du+\int_{3}^{x} 0\,du$$

$$= 0+\left[\frac{(u-1)^2}{2}\right]_1^2+\left[-\frac{(3-u)^2}{2}\right]_2^3+0$$

$$= \frac{1}{2}+\frac{1}{2}=1.$$

These give $F(x)=\begin{cases}0, & -\infty < x < 1\\ \dfrac{(x-1)^2}{2}, & 1\le x<2\\ 1-\dfrac{(3-x)^2}{2}, & 2\le x<3\\ 1 & 3\le x<\infty\end{cases}$



Distribution function

Fig. 11.17

(ii) $P(1.5\le X\le 2.5) = F(2.5)-F(1.5)$

$$=\left(1-\frac{(3-2.5)^2}{2}\right)-\left(\frac{(1.5-1)^2}{2}\right)$$

$$=\frac{1.75-0.25}{2}=0.75$$

or

$$P(1.5\le X\le 2.5) = \int_{1.5}^{2.5} f(x)\,dx=\int_{1.5}^{2}(x-1)\,dx+\int_{2}^{2.5}(-x+3)\,dx=0.75.$$

Check: (i) Whether $F(x)$ is continuous everywhere.

(ii) From the Fig. 11.16, triangle area $=\frac{1}{2}bh=1$.

11.4.5 Probability density function from Probability distribution function.

Let us learn the method to determine the probability density function $f(x)$ from the distribution function $F(x)$ of a continuous random variable X.

Suppose $F(x)$ is the distribution function of a continuous random variable X. Then the probability density function $f(x)$ is given by

$$f(x)=\frac{dF(x)}{dx}=F'(x),\ \textit{wherever derivative exists.}$$

Example 11.13

If X is the random variable with distribution function $F(x)$ given by,

$$F(x)=\begin{cases}0, & x<0\\ x, & 0\le x<1\\ 1, & 1\le x\end{cases}$$

then find (i) the probability density function $f(x)$ (ii) $P(0.2\le X\le 0.7)$.

Solution

(i) Differentiating $F(x)$ with respect to x at continuity points of $f(x)$, we get

$$f(x) = F'(x) = \begin{cases} 0, & x < 0 \\ 1, & 0 \le x < 1 \\ 0, & x \ge 1 \end{cases}$$

The pdf $f(x)$ is not continuous at $x = 0$, or at $x = 1$. We can define $f(0)$ and $f(1)$ in any manner. Choosing $f(0) = 1$, and $f(1) = 0$.

Therefore the probability density function $f(x)$ is

$$f(x) = \begin{cases} 1, & 0 \le x < 1 \\ 0, & \text{otherwise} \end{cases}$$

(ii) $$P(0.2 \le X \le 0.7) = F(0.7) - F(0.2)$$

$$= 0.7 - 0.2 = 0.5$$

or

$$P(0.2 \le X \le 0.7) = \int_{0.2}^{0.7} f(x)\,dx = \int_{0.2}^{0.7} 1\,dx = 0.5 .$$

Remark

By definition, $P(X \le x) = F(x) = \int_{-\infty}^{x} f(u)\,du$. Probability $P(a < X < b)$ can be obtained by using either $F(x)$ or $f(x)$.

Note

We may also define the above probability density function as

$$f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases} \quad \text{or} \quad f(x) = \begin{cases} 1, & 0 < x \le 1 \\ 0, & \text{otherwise} \end{cases} \quad \text{or} \quad f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Example 11.14

The probability density function of random variable X is given by $f(x) = \begin{cases} k & 1 \le x \le 5 \\ 0 & \text{otherwise} \end{cases}$

Find (i) Distribution function (ii) $P(X < 3)$ (iii) $P(2 < X < 4)$ (iv) $P(3 \le X)$

Solution

Since $f(x)$ is a probability density function, $f(x) \ge 0$ and $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

That is $$\int_{-\infty}^{1} 0\,dx + \int_{1}^{5} k\,dx + \int_{5}^{\infty} 0\,dx = 1 .$$

$$0 + k(x)_1^5 + 0 = 1 \Rightarrow 4k = 1 \Rightarrow k = \frac{1}{4} .$$

Therefore the probability density function is

$$f(x) = \begin{cases} \frac{1}{4}, & 1 \le x \le 5 \\ 0, & \text{otherwise} \end{cases}$$

(i) Distribution function

The distribution function $F(x) = P(X \le x) = \int_{-\infty}^{x} f(u)\,du$.

When $x < 1$, $F(x) = \int_{-\infty}^{x} f(u)\,du = \int_{-\infty}^{x} 0\,du = 0$.

When $1 \le x < 5$ $F(x) = \int_{-\infty}^{x} f(u)\,du = \int_{-\infty}^{1} 0\,du + \int_{1}^{x} \frac{1}{4}\,du = \frac{1}{4}(x-1)$.

When $x \ge 5$ $F(x) = \int_{-\infty}^{x} f(u)\,du = \int_{-\infty}^{1} 0\,du + \int_{1}^{5} \frac{1}{4}\,du + \int_{5}^{x} 0\,du = 1$.

Thus $F(x) = \begin{cases} 0, & x < 1 \\ \frac{x-1}{4}, & 1 \le x \le 5 \\ 1, & x > 5. \end{cases}$

(ii) $P(X < 3) = P(X \le 3) = F(3) = \frac{3-1}{4} = \frac{1}{2}$ (since $F(x)$ is continuous).

(iii) $P(2 < X < 4) = P(2 \le X \le 4) = F(4) - F(2) = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$.

(iv) $P(3 \le X) = P(X \ge 3) = 1 - P(X < 3) = 1 - \frac{1}{2} = \frac{1}{2}$.

Example 11.15

Let X be a random variable denoting the life time of an electrical equipment having probability density function

$$f(x) = \begin{cases} k e^{-2x} & \text{for } x > 0 \\ 0 & \text{for } x \le 0. \end{cases}$$

Find (i) the value of k (ii) Distribution function (iii) $P(X < 2)$

(iv) calculate the probability that X is at least for four unit of time (v) $P(X = 3)$.

Solution

(i) Since $f(x)$ is a probability density function, $f(x) \ge 0$ and $\int_{-\infty}^{\infty} f(x)\,dx = 1$

That is $\int_{-\infty}^{0} 0\,dx + \int_{0}^{\infty} k e^{-2x}\,dx = 1$.

$$0 + k\left(\frac{e^{-2x}}{-2}\right)_0^{\infty} = 1 \Rightarrow k\left(\frac{e^{-\infty} - e^{0}}{-2}\right) = 1 \Rightarrow k = 2 .$$

Therefore the probability density function is

$$f(x) = \begin{cases} 2e^{-2x} & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \end{cases}$$

(ii) Distribution function

By definition the distribution function $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(u)\,du$

When $x \leq 0$ $\quad F(x) = \int_{-\infty}^{x} f(u)\,du = \int_{-\infty}^{x} 0\,du = 0$.

When $x > 0$ $\quad F(x) = \int_{-\infty}^{x} f(u)\,du = \int_{-\infty}^{0} 0du + \int_{0}^{x} 2e^{-2u}du = 2\left(\frac{e^{-2u}}{-2}\right)_0^x = 1 - e^{-2x}$.

This gives $\quad F(x) = \begin{cases} 0, & \text{for } x \leq 0 \\ 1 - e^{-2x} & \text{for } x > 0 \end{cases}$.

(iii) $P(X < 2) = P(X \leq 2) = F(2) = 1 - e^{-2\times 2} = 1 - e^{-4}$ (since $F(x)$ is continuous).

(iv) The probability that X is at least equal to four unit of time is
$P(X \geq 4) = 1 - P(X < 4) = 1 - F(4) = 1 - (1 - e^{-2\times 4}) = e^{-8}$.

(v) In the continuous case, $f(x)$ at $x = a$ is not the probability that X takes the value a, that is $f(x)$ at $x = a$ is not equal to $P(X = a)$. If X is continuous type, $P(X = a) = 0$ for $a \in \mathbb{R}$. Therefore $P(X = 3) = 0$.

EXERCISE11.3

1. The probability density function of X is given by $f(x) = \begin{cases} kxe^{-2x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$.

   Find the value of k.

2. The probability density function of X is $f(x) = \begin{cases} x & 0 < x < 1 \\ 2 - x & 1 \leq x < 2 \\ 0 & \text{otherwise} \end{cases}$.

   Find (i) $P(0.2 \leq X < 0.6)$ (ii) $P(1.2 \leq X < 1.8)$ (iii) $P(0.5 \leq X < 1.5)$

3. Suppose the amount of milk sold daily at a milk booth is distributed with a minimum of 200 litres and a maximum of 600 litres with probability density function

   $f(x) = \begin{cases} k & 200 \leq x \leq 600 \\ 0 & \text{otherwise} \end{cases}$

   Find (i) the value of k (ii) the distribution function

   (iii) the probability that daily sales will fall between 300 litres and 500 litres?

4. The probability density function of X is given by $f(x) = \begin{cases} ke^{-\frac{x}{3}} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$

   Find (i) the value of k (ii) the distribution function (iii) $P(X < 3)$

   (iv) $P(5 \leq X)$ (v) $P(X \leq 4)$.

5. If X is the random variable with probability density function $f(x)$ given by,

$$f(x)=\begin{cases} x+1, & -1\le x<0 \\ -x+1, & 0\le x<1 \\ 0 & \text{otherwise} \end{cases}$$

then find (i) the distribution function $F(x)$ (ii) $P(-0.5\le X\le 0.5)$

6. If X is the random variable with distribution function $F(x)$ given by,

$$F(x)=\begin{cases} 0, & -\infty< x< 0 \\ \frac{1}{2}\left(x^2+x\right) & 0\le x<1 \\ 1, & 1\le x<\infty \end{cases}$$

then find (i) the probability density function $f(x)$ (ii) $P(0.3\le X\le 0.6)$

11.5 Mathematical Expectation

One of the important characteristics of a random variable is its expectation. Synonyms for expectation are expected value, mean, and first moment.

The definition of mathematical expectation is driven by conventional idea of numerical average. The numerical average of n numbers, say $a_1, a_2, a_3, \ldots, a_n$ is

$$\frac{a_1+a_2+a_3+\ldots+a_n}{n}.$$

The average is used to summarize or characterize the entire collection of n numbers $a_1, a_2, a_3, \ldots, a_n$, with single value.

Illustration 11.7

Consider ten numbers $6, 2, 5, 5, 2, 6, 2, -4, 1, 5$.

The average is $\dfrac{6+2+5+5+2+6+2-4+1+5}{10}=3$.

If ten numbers $6, 2, 5, 5, 2, 6, 2, -4, 1, 5$ are considered as the values of a random variable X the probability mass function is given by

x	-4	1	2	5	6
$P(X=x)$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{3}{10}$	$\frac{2}{10}$

The above calculation for average can also be rewritten as

$$-4\times\frac{1}{10}+1\times\frac{1}{10}+2\times\frac{3}{10}+5\times\frac{3}{10}+6\times\frac{2}{10}=3.$$

This illustration suggests that the mean or expected value of any random variable may be obtained by the sum of the product of each value of the random variable by its corresponding probability.

So average $=\sum$ (value of x) $\times$ (probability)

This is true if the random variable is discrete. In the case of continuous random variable, the mathematical expectation is essentially the same with summations being replaced by integrals.

Two quantities are often used to summarize a probability distribution of a random variable X. In terms of statistics one is central tendency and the other is dispersion or variability of the probability distribution. The mean is a measure of the centre tendency of the probability distribution, and the variance is a measure of the dispersion, or variability in the distribution. But these two measures do not uniquely identify a probability distribution. That is, two different distributions can have the same mean and variance. Still, these measures are simple, and useful in the study of the probability distribution of X.

11.5.1 Mean

Definition 11.8 : (Mean)

*Suppose X is a random variable with probability mass (or) density function $f(x)$. **The expected value or mean or mathematical expectation of** X*, denoted by $E(X)$ or μ *is*

$$E(X)=\begin{cases}\sum\limits_{x} x\,f(x) & \text{if } X \text{ is discrete}\\ \int\limits_{-\infty}^{\infty} x\,f(x)\,dx & \text{if } X \text{ is continuous}\end{cases}$$

The expected value is in general not a typical value that the random variable can take on. It is often helpful to interpret the expected value of a random variable as the long-run average value of the variable over many independent repetitions of an experiment.

Theorem 11.3 (Without proof)

*Suppose X is a random variable with probability mass (or) density function $f(x)$. **The expected value of the function** $g(X)$, a new random variable is*

$$E(g(X))=\begin{cases}\sum\limits_{x} g(x)\,f(x) & \text{if } g(x) \text{ is discrete}\\ \int\limits_{-\infty}^{\infty} g(x)\,f(x)\,dx & \text{if } g(x) \text{ is continuous}\end{cases}$$

If $g(X)=x^k$ the above theorem yield the expected value called *the k-th moment about the origin* of the random variable X.

Therefore the k-th moment about the origin of the random variable X is

$$E(X^k)=\begin{cases}\sum\limits_{x} x^k\,f(x) & \text{if } X \text{ is discrete}\\ \int\limits_{-\infty}^{\infty} x^k\,f(x)\,dx & \text{if } X \text{ is continuous}\end{cases}$$

Note

When $k=0$, by definition,

$$E(1)=\begin{cases}\sum_{x} f(x)=1 & \text{if } X \text{ is discrete}\\ \int_{-\infty}^{\infty} f(x)\,dx=1 & \text{if } X \text{ is continuous}\end{cases}$$

11.5.2 Variance

Variance is a statistical measure that tells us how measured data vary from the average value of the set of data. Mathematically, variance is the mean of the squares of the deviations from the arithmetic mean of a data set. The terms variability, spread, and dispersion are synonyms, and refer to how spread out a distribution is.

Definition 11.9: (Variance)

*The **variance** of a random variable X* denoted by $\text{Var}(X)$ or $V(X)$ or σ^2 (or $\sigma_x^{\,2}$) is

$$V(X)=E(X-E(X))^2=E(X-\mu)^2$$

Square root of variance is called standard deviation. That is standard deviation $\sigma=\sqrt{V(X)}$. The variance and standard deviation of a random variable are always non negative.

11.5.3 Properties of Mathematical expectation and variance

(i) $E(aX+b)=aE(X)+b$, where a and b are constants

Proof

Let X be a discrete random variable

$$E(aX+b)=\sum_{i=1}^{\infty}(ax_i+b)f(x_i) \qquad \text{(by definition)}$$

$$=\sum_{i=1}^{\infty}\left(ax_i f(x_i)+bf(x_i)\right)$$

$$=a\sum_{i=1}^{\infty}x_i f(x_i)+b\sum_{i=1}^{\infty}f(x_i)$$

$$=aE(X)+b(1) \qquad \left(\because \sum_{i=1}^{\infty}f(x_i)=1\right)$$

$$E(aX+b)=aE(X)+b.$$

■

Similarly, when X is a continuous random variable, we can prove it, by replacing summation by integration.

Corollary 1:	$E(aX)=aE(X)$	(when $b=0$)
Corollary 2:	$E(b)=b$	(when $a=0$)

(ii) $Var(X) = E(X^2) - \left(E(X)\right)^2$

Proof

$$\begin{aligned} \text{We know} \quad E(x) &= \mu \\ Var(X) &= E(X-\mu)^2 \\ &= E(X^2 - 2X\mu + \mu^2) \\ &= E(X^2) - 2\mu E\left(X\right) + \mu^2 \qquad \text{(Since } \mu \text{ is a constant)} \\ &= E(X^2) - 2\mu\mu + \mu^2 = E(X^2) - \mu^2 \\ Var(X) &= E(X^2) - \left(E(X)\right)^2 \end{aligned}$$

An alternative formula to compute variance of a random variable X is

$$\sigma^2 = \text{Var}(X) = E(X^2) - \left(E(X)\right)^2$$

■

(iii) $Var(aX+b) = a^2 Var(X)$ where a and b are constants

Proof

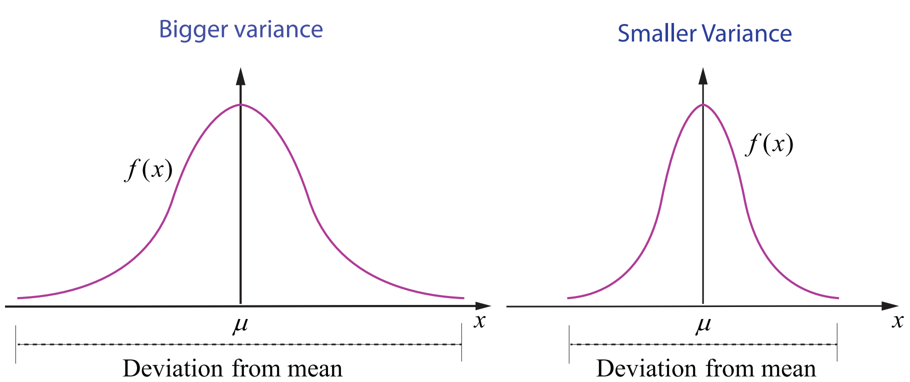
$$\begin{aligned} Var(aX+b) &= E\left((aX+b) - E(aX+b)\right)^2 \\ &= E\left(aX + b - aE(X) - b\right)^2 \\ &= E\left(aX - aE(X)\right)^2 \\ &= E\left(a\left(X - E(X)\right)\right)^2 \\ &= a^2 E\left(X - E(X)\right)^2. \end{aligned}$$

■

$$\text{Hence} \quad Var(aX+b) = a^2 Var(X)$$

Corollary 3:	$V(aX) = a^2 V(X)$	(when $b = 0$)
Corollary 4:	$V(b) = 0$	(when $a = 0$)

Variance gives information about the deviation of the values of the random variable about the mean μ. A smaller σ^2 implies that the random values are more clustered about the mean, similarly, a bigger σ^2 implies that the random values are more scattered from the mean.



Different variance with same mean

Fig. 11.18

The above figure shows the pdfs of two continuous random variables whose curves are bell-shaped with same mean but different variances. ■

Example 11.16

Suppose that $f(x)$ given below represents a probability mass function,

x	1	2	3	4	5	6
$f(x)$	c^2	$2c^2$	$3c^2$	$4c^2$	c	$2c$

Find (i) the value of c (ii) Mean and variance.

Solution

(i) Since $f(x)$ is a probability mass function, $f(x) \geq 0$ for all x, and $\sum_x f(x) = 1$.

$$\text{Thus, } \sum_x f(x) = 1$$

$$c^2 + 2c^2 + 3c^2 + 4c^2 + c + 2c = 1$$

$$c = \frac{1}{5} \text{ or } -\frac{1}{2}.$$

Since $f(x) \geq 0$ for all x, the possible value of c is $\frac{1}{5}$.

Hence, the probability mass function is

x	1	2	3	4	5	6
$f(x)$	$\frac{1}{25}$	$\frac{2}{25}$	$\frac{3}{25}$	$\frac{4}{25}$	$\frac{1}{5}$	$\frac{2}{5}$

(ii) To find mean and variance, let us use the following table

x	$f(x)$	$x\,f(x)$	$x^2 f(x)$
1	$\frac{1}{25}$	$\frac{1}{25}$	$\frac{1}{25}$
2	$\frac{2}{25}$	$\frac{4}{25}$	$\frac{8}{25}$
3	$\frac{3}{25}$	$\frac{9}{25}$	$\frac{27}{25}$
4	$\frac{4}{25}$	$\frac{16}{25}$	$\frac{64}{25}$
5	$\frac{1}{5}$	$\frac{5}{5}$	$\frac{25}{5}$
6	$\frac{2}{5}$	$\frac{12}{5}$	$\frac{72}{5}$
	$\sum f(x) = 1$	$\sum x\,f(x) = \frac{115}{25}$	$\sum x^2 f(x) = \frac{585}{25}$

Mean : $E(X) = \sum x\, f(x) = \frac{115}{25} = 4.6$

Variance : $V(X) = E(X^2) - (E(X))^2 = \sum x^2 f(x) - \left(\sum x\, f(x)\right)^2$

$$= \frac{585}{25} - \left(\frac{115}{25}\right)^2 = 23.40 - 21.16 = 2.24$$

Therefore the mean and variance are 4.6 and 2.24 respectively.

Example 11.17

Two balls are chosen randomly from an urn containing 8 white and 4 black balls. Suppose that we win Rs 20 for each black ball selected and we lose Rs10 for each white ball selected. Find the expected winning amount and variance.

Solution

Let X denote the winning amount. The possible events of selection are (i) both balls are black, or (ii) one white and one black or (iii) both are white. Therefore X is a random variable that can be defined as

X (both are black balls) = ₹ 2(20) = ₹ 40

X (one black and one white ball) = ₹ 20 – ₹ 10 = ₹10

X (both are white balls) = ₹ (–20) = – ₹ 20

Therefore X takes on the values 40, 10 and –20

Total number of balls $n = 12$

Total number of ways of selecting 2 balls $= \binom{12}{2} = \frac{12 \times 11}{1 \times 2} = 66$

Number of ways of selecting 2 black balls $= \binom{4}{2} = 6$

Number of ways of selecting one black ball and one white ball $= \binom{8}{1}\binom{4}{1} = 32$

Number of ways of selecting 2 white balls $= \binom{8}{2} = 28$

Values of Random Variable X	40	10	–20	Total
Number of elements in inverse images	6	32	28	66

Probability mass function is

X	40	10	–20	Total
$f(x)$	$\frac{6}{66}$	$\frac{32}{66}$	$\frac{28}{66}$	1

Mean :

$$E(X) = \sum x\, f(x) = 40 \cdot \left(\frac{6}{66}\right) + 10 \cdot \left(\frac{32}{66}\right) + (-20) \cdot \left(\frac{28}{66}\right) = 0$$

That is expected winning amount is 0.

Variance :

$$E(X^2) = \sum x^2\, f(x) = 40^2 \cdot \left(\frac{6}{66}\right) + 10^2 \cdot \left(\frac{32}{66}\right) + (-20)^2 \cdot \left(\frac{28}{66}\right) = \frac{4000}{11}$$

$$(E(X))^2 = 0^2 = 0$$

This gives $V(X) = E(X^2) - (E(X))^2 = \frac{4000}{11} - 0 = \frac{4000}{11}$

Therefore $E(X) = 0$ and $\text{Var}(X) = \frac{4000}{11}$.

Example 11.18

Find the mean and variance of a random variable X, whose probability density function is

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Solution

Observe that the given distribution is continuous

Mean :

By definition $\mu = E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$

$$= \int_{-\infty}^{0} 0\left(\lambda e^{-\lambda x}\right) dx + \int_{0}^{\infty} x\left(\lambda e^{-\lambda x}\right) dx$$

(We can also use integration by parts or Bernoulli's formula)

$$= 0 + \lambda \int_{0}^{\infty} x\left(e^{-\lambda x}\right) dx$$

$$= 0 + \lambda \left(\frac{\lfloor 1}{\lambda^2}\right) \quad \left(\text{using Gamma integral for positive integer } n,\ \int_{0}^{\infty} x^n e^{-\alpha x} dx = \frac{\lfloor n}{\alpha^{n+1}}\right)$$

$$= \frac{1}{\lambda}$$

Variance :

By definition, $E(X^2) = \int_{-\infty}^{\infty} x^2\, f(x)\, dx$

$$= \int_{-\infty}^{0} 0\left(\lambda e^{-\lambda x}\right) dx + \int_{0}^{\infty} x^2\left(\lambda e^{-\lambda x}\right) dx$$

(We can also use integration by parts or Bernoulli's formula)

$$= 0 + \lambda \int_{0}^{\infty} x^2\left(e^{-\lambda x}\right) dx$$

$$= 0 + \lambda \left(\frac{\lfloor 2}{\lambda^3}\right) = \frac{2}{\lambda^2} \quad \text{(using Gamma integral for positive integer)}$$

Therefore $Var(X) = E(X^2) - (E(X))^2$

$$= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

Hence the mean and variance are respectively $\frac{1}{\lambda}$ and $\frac{1}{\lambda^2}$. ■

EXERCISE 11.4

1. For the random variable X with the given probability mass function as below, find the mean and variance.

(i) $f(x) = \begin{cases} \frac{1}{10} & x = 2, 5 \\ \frac{1}{5} & x = 0, 1, 3, 4 \end{cases}$ (ii) $f(x) = \begin{cases} \frac{4-x}{6} & x = 1, 2, 3 \end{cases}$

(iii) $f(x) = \begin{cases} 2(x-1) & 1 < x << 2 \\ 0 & \text{otherwise} \end{cases}$ (iv) $f(x) = \begin{cases} \frac{1}{2} e^{-\frac{x}{2}} & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$

2. Two balls are drawn in succession without replacement from an urn containing four red balls and three black balls. Let X be the possible outcomes drawing red balls. Find the probability mass function and mean for X.
3. If μ and σ^2 are the mean and variance of the discrete random variable X, and $E(X+3) = 10$ and $E(X+3)^2 = 116$, find μ and σ^2.
4. Four fair coins are tossed once. Find the probability mass function, mean and variance for number of heads occurred.
5. A commuter train arrives punctually at a station every half hour. Each morning, a student leaves his house to the train station. Let X denote the amount of time, in minutes, that the student waits for the train from the time he reaches the train station. It is known that the pdf of X is

$$f(x) = \begin{cases} \frac{1}{30} & 0 < x < 30 \\ 0 & \text{elsewhere} \end{cases}$$

Obtain and interpret the expected value of the random variable X.

6. The time to failure in thousands of hours of an electronic equipment used in a manufactured computer has the density function

$$f(x) = \begin{cases} 3e^{-3x} & x > 0 \\ 0 & \text{elsewhere} \end{cases}.$$

Find the expected life of this electronic equipment.

7. The probability density function of the random variable X is given by

$$f(x) = \begin{cases} 16xe^{-4x} & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \end{cases}$$

find the mean and variance of X.

8. A lottery with 600 tickets gives one prize of ₹200, four prizes of ₹100, and six prizes of ₹ 50. If the ticket costs is ₹ 2, find the expected winning amount of a ticket.

11.6 Theoretical Distributions: Some Special Discrete Distributions

In the previous section we have dealt with various general probability distributions with mean and variance. We shall now learn some discrete probability distributions of special importance.

In this section we learn the following discrete distributions.

(i) The One point distribution (ii) The Two point distribution

(iii) The Bernoulli distribution (iv) The Binomial distribution.

11.6.1 The One point distribution

The random variable X has a one point distribution if there exists a point x_0 such that, the probability mass function $f(x)$ is defined as $f(x) = P(X = x_0) = 1$.

That is the probability mass is concentrated at one point.

The cumulative distribution function is

$$F(x) = \begin{cases} 0 & -\infty < x < x_0 \\ 1 & x_0 \le x < \infty \end{cases}$$

Mean :

$$E(X) = \sum_x x\,f(x) = x_0 \times 1 = x_0$$

Variance :

$$V(X) = E(X^2) - \left(E(X)\right)^2 = \sum_x x^2 f(x) - \left(x_0\right)^2 = x_0^{\,2} - x_0^{\,2} = 0$$

Therefore the mean and the variance are respectively x_0 and 0.

11.6.2 The Two point distribution

(a) Unsymmetrical Case: The random variable X has a two point distribution if there exists two values x_1 and x_2, such that

$$f(x) = \begin{cases} p & \text{for } x = x_1 \\ 1-p & \text{for } x = x_2 \end{cases} \qquad \text{where } 0 < p < 1.$$

The cumulative distribution function is

$$F(x) = \begin{cases} 0 & \text{if } x < x_1 \\ p & \text{if } x_1 \le x < x_2 \\ 1 & \text{if } x \ge x_2 \end{cases}$$

Mean :

$$E(X) = \sum_x x\,f(x) = x_1 \times p + x_2 \times (1-p) = px_1 + qx_2 \text{ where } q = 1-p.$$

Variance :

$$\begin{aligned} V(X) &= E(X^2) - \left(E(X)\right)^2 = \sum_x x^2 f(x) - \left(px_1 + qx_2\right)^2 \\ &= \left(x_1^{\,2} p + x_2^{\,2} q\right) - \left(px_1 + qx_2\right)^2 = pq\left(x_2 - x_1\right)^2 \end{aligned}$$

The mean and the variance are respectively $px_1 + qx_2$ and $pq\left(x_2 - x_1\right)^2$

(b) Symmetrical Case:

When $p = q = \frac{1}{2}$, the two point distribution becomes

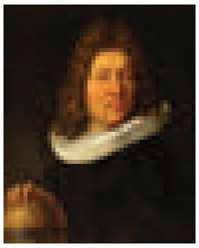
$$f(x) = \begin{cases} \frac{1}{2} & \text{for } x = x_1 \\ \frac{1}{2} & \text{for } x = x_2 \end{cases}$$

and the cumulative distribution function is

$$F(x) = \begin{cases} 0 & \text{if } x < x_1 \\ \frac{1}{2} & \text{if } x_1 \le x < x_2 \\ 1 & \text{if } x \ge x_2 \end{cases}$$

The mean and variance respectively are $\frac{x_1 + x_2}{2}$ and $\frac{(x_2 - x_1)^2}{4}$.

11.6.3 The Bernoulli distribution



Jacob Bernoulli (1654 - 1705)

Independent trials having constant probability of success p were first studied by the Swiss mathematician Jacques Bernoulli (1654–1705). In his book *The Art of Conjecturing*, published by his nephew Nicholas eight years after his death in 1713, Bernoulli showed that if the number of such trials were large, then the proportion of them that were successes would be close to p.

In **probability theory,** the **Bernoulli distribution**, named after Swiss mathematician **Jacob Bernoulli** is the **discrete probability distribution** of a **random variable**. A Bernoulli experiment is a random experiment, where the outcomes is classified in one of two mutually exclusive and exhaustive ways, say success or failure (example: heads or tails, defective item or good item, life or death or many other possible pairs). A sequence of Bernoulli trails occurs when a Bernoulli experiment is performed several independent times so that the probability of success remains the same from trial to trial. Any nontrivial experiment can be dichotomized to yield Bernoulli model.

Definition 11.10: (Bernoulli's distribution)

Let X be a random variable associated with a Bernoulli trial by defining it as X (success) = 1 and X (failure) = 0, such that

$$f(x) = \begin{cases} p & x = 1 \\ q = 1 - p & x = 0 \end{cases} \qquad \text{where } 0 < p < 1 ,$$

then X is called a Bernoulli random variable and $f(x)$ is called the Bernoulli distribution.

Or equivalently

If a random variable X is following a Bernoulli's distribution, with probability p of success can be denoted as $X \sim Ber(p)$, where p is called the parameter, then the probability mass function of X is

$$f(x) = p^x (1 - p)^{1-x}, \qquad x = 0, 1$$

The cumulative distribution of Bernoulli's distribution is

$$F(x)=\begin{cases}0 & \text{if } \ x<0\\ q=1-p & \text{if } \ 0\le x<1\\ 1 & \text{if } \ x\ge 1\end{cases}$$

Mean :

$$E(X) = \sum_x x\,f(x)=1\times p+0\times(1-p)=p\,,$$

Note that, since X takes only the values 0 and 1, its expected value p is "never seen".

Variance :

$$V(X) = E(X^2)-\left(E(X)\right)^2=\sum_x x^2 f(x)-p^2$$
$$= \left(1^2 p+0^2 q\right)-p^2 = p(1-p) = pq \quad \text{where } q=1-p$$

> *If X* is a Bernoulli's random variable which follows Bernoulli distribution with parameter p, the mean μ and variance σ^2 are
>
> $$\mu = p \qquad \text{and} \qquad \sigma^2 = pq$$

When $p=q=\dfrac{1}{2}$, the Bernoulli's distribution become

$$f(x)=\begin{cases}\dfrac{1}{2} & \text{for } x=0\\[2mm] \dfrac{1}{2} & \text{for } x=1\end{cases}$$

and the cumulative distribution is

$$F(x)=\begin{cases}0 & \text{if } \ x<0\\ \dfrac{1}{2} & \text{if } 0\le x<1\\ 1 & \text{if } \ x\ge 1\end{cases}$$

The mean and variance are respectively are $\dfrac{1}{2}$ and $\dfrac{1}{4}$

11.6.4 The Binomial Distribution

The Binomial Distribution is an important distribution which applies in some cases for repeated independent trials where there are only two possible outcomes: heads or tails, success or failure, defective item or good item, or many other such possible pairs. The probability of each outcome can be calculated using the multiplication rule, perhaps with a tree diagram.

Suppose a coin is tossed once. Let X denote the number of heads. Then $X \sim Ber(p)$, because we get either head $(X=1)$ or tail $(X=0)$ with probability p or $1-p$.

Suppose a coin is tossed n times. Let X denote the number of heads. Then X takes on the values 0, 1, 2, …, n. The probability for getting x number of heads is given by

$$P(X=x)=\binom{n}{x}p^x(1-p)^{n-x}\,,\ \ x=0,\,1,\,2,\,...,\,n.$$

$X = x$, corresponds to the combination of x heads in n tosses, that is $\binom{n}{x}$ ways of heads and remaining $n-x$ tails. Hence, the probability for each of those outcomes is equal to $p^x(1-p)^{n-x}$. Binomial theorem is suitable to apply when n is small number less than 30.

Definition 11.11: Binomial random variable

A discrete random variable X is called binomial random variable, if X is the number of successes in n-repeated trials such that

(i) the n- repeated trials are independent and n is finite
(ii) each trial results only two possible outcomes, labelled as 'success' or 'failure'
(iii) the probability of a success in each trial, denoted as p, remains constant.

Definition 11.12 : Binomial distribution

The binomial random variable X equals the number of successes with probability p for a success and $q = 1-p$ for a failure in n-independent trials, has ***a binomial distribution*** denoted by $X \sim B(n, p)$. The probability mass function of X is

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}, \qquad x = 0,1,2,\ldots,n.$$

The name of the distribution is obtained from the *binomial expansion*. For constants a and b, the binomial expansion is

$$(a+b)^n = \sum_{x=0}^{n} \binom{n}{x} a^x b^{n-x}$$

Let p denote the probability of success on a single trial. Then, by using the binomial expansion with $a = p$ and $b = 1-p$, we see that the sum of the probabilities for a binomial random variable is 1. Since each trial in the experiment is classified into two outcomes, {success, failure}, the distribution is called a "bi"-nomial.

If X is a binomial random variable which follows binomial distribution with parameters p and n, the mean μ and variance σ^2 are

$$\mu = np \quad \text{and} \quad \sigma^2 = np(1-p)$$

The expected value is in general not a typical value that the random variable can take on. It is often helpful to interpret the expected value of a random variable as the long-run average value of the variable over many independent repetitions of an experiment. The shape of a **binomial distribution** is **symmetrical** when $p = 0.5$ or when n is large.

When $p = q = \frac{1}{2}$, the binomial distribution becomes

$$f(x) = \binom{n}{x}\left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{n-x}, \qquad x = 0,1,2,\ldots,n.$$

That is

$$f(x) = \binom{n}{x}\left(\frac{1}{2}\right)^n, \qquad x = 0,1,2,\ldots,n.$$

The mean and variance are respectively are $\frac{n}{2}$ and $\frac{n}{4}$

Example 11.19

Find the binomial distribution for each of the following.

(i) Five fair coins are tossed once and X denotes the number of heads.

(ii) A fair die is rolled 10 times and X denotes the number of times 4 appeared.

Solution

(i) Given that five fair coins are tossed once. Since the coins are fair coins the probability of getting an head in a single coin is $p = \frac{1}{2}$ and $q = 1 - p = \frac{1}{2}$

Let X denote the number of heads that appear in five coins. X is a binomial random variable that takes on the values 0, 1,2,3,4 and 5, with $n = 5$ and $p = \frac{1}{2}$.That is $X \sim B\left(5, \frac{1}{2}\right)$.

Therefore the binomial distribution is

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}, \qquad x = 0,1,2,\ldots,n$$

becomes

$$f(x) = \binom{5}{x}\left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{n-x}, \qquad x = 0,1,2,\ldots,5.$$

That is
$$f(x) = \binom{5}{x}\left(\frac{1}{2}\right)^n, \qquad x = 0,1,2,\ldots,n$$

(ii) A fair die is rolled ten times and X denotes the number of times 4 appeared. X is binomial random variable that takes on the values $0, 1, 2, 3, \ldots 10$, with $n = 10$ and $p = \frac{1}{6}$. That is $X \sim B\left(10, \frac{1}{6}\right)$.

Probability of getting a four in a die is $p = \frac{1}{6}$ and $q = 1 - p = \frac{5}{6}$.

Therefore the binomial distribution is

$$f(x) = \binom{10}{x}\left(\frac{1}{6}\right)^x \left(\frac{5}{6}\right)^{10-x}, \qquad x = 0,1,2,\ldots,10.$$

Example 11.20

A multiple choice examination has ten questions, each question has four distractors with exactly one correct answer. Suppose a student answers by guessing and if X denotes the number of correct answers, find (i) binomial distribution (ii) probability that the student will get seven correct answers (iii) the probability of getting at least one correct answer.

Solution

(i) Since X denotes the number of success, X can take the values $0,1,2,...,10$

The probability for success is $p=\frac{1}{4}$ and for failure $q=1-p=\frac{3}{4}$, and $n=10$.

Therefore X follows the binomial distribution $X \sim B\left(10,\frac{1}{4}\right)$.

This gives,
$$f(x) = \binom{10}{x}\left(\frac{1}{4}\right)^{x}\left(\frac{3}{4}\right)^{10-x}, \qquad x=0,1,2,...,10$$

(ii) Probability for seven correct answers is
$$P(X=7) = f(7)=\binom{10}{7}\left(\frac{1}{4}\right)^{7}\left(\frac{3}{4}\right)^{10-7} =120\left(\frac{3^3}{4^{10}}\right)$$

Probability that the student will get seven correct answers is $120\left(\frac{3^3}{4^{10}}\right)$.

(iii) Probability for at least one correct answer is
$$P(X\geq 1) = 1-P(X<1)=1-P(X=0)$$
$$=1-\binom{10}{0}\left(\frac{1}{4}\right)^{0}\left(\frac{3}{4}\right)^{10} = 1-\left(\frac{3}{4}\right)^{10}.$$

Probability that the student will get for at least one correct answer is $1-\left(\frac{3}{4}\right)^{10}$.

Example 11.21

The mean and variance of a binomial variate X are respectively 2 and 1.5. Find

(i) $P(X=0)$ (ii) $P(X=1)$ (iii) $P(X\geq 1)$

Solution

To find the probabilities, the values of the parameters n and p must be known.

Given that
$$\text{Mean} = np=2 \text{ and variance } =npq=1.5$$

This gives $\frac{npq}{np} = \frac{1.5}{2}=\frac{3}{4}$
$$q = \frac{3}{4} \text{ and } p=1-q=1-\frac{3}{4}=\frac{1}{4}$$
$$np = 2\text{, gives } n=\frac{2}{p}=8\text{. Therefore } X\sim B\left(8,\frac{1}{4}\right).$$

Therefore the binomial distribution is
$$P(X=x) = f(x)=\binom{8}{x}\left(\frac{1}{4}\right)^{x}\left(\frac{3}{4}\right)^{8-x} \qquad x=0,1.2,...8$$

(i)
$$P(X=0) = f(0)=\binom{8}{0}\left(\frac{1}{4}\right)^{0}\left(\frac{3}{4}\right)^{8-0} =\left(\frac{3}{4}\right)^{8}$$

(ii) $$P(X=1) = f(1) = \binom{8}{1}\left(\frac{1}{4}\right)^{1}\left(\frac{3}{4}\right)^{8-1} = 2\left(\frac{3}{4}\right)^{7}$$

(iii) $$P(X \geq 1) = 1 - P(X<1) = 1 - P(X=0) = 1 - \left(\frac{3}{4}\right)^{8}$$

Example 11.22

On the average, 20% of the products manufactured by ABC Company are found to be defective. If we select 6 of these products at random and X denotes the number of defective products find the probability that (i) two products are defective (ii) at most one product is defective (iii) at least two products are defective.

Solution

Given that $n=6$

Probability for selecting a defective product is $\frac{20}{100}$, that is $p=\frac{1}{5}$.

Since X denotes the number defective products, X can take on the values $0,1,2,...,6$

The probability for defective (success) is $p=\frac{1}{5}$ and for failure $q=1-p=\frac{4}{5}$, and $n=6$

Therefore X follows the binomial distribution denoted by $X \sim B\left(6,\frac{1}{5}\right)$.

This gives $$f(x) = \binom{6}{x}\left(\frac{1}{5}\right)^{x}\left(\frac{4}{5}\right)^{6-x}, \quad x=0,1,2,...,6.$$

R7C1C7

(i) Probability for two defective products is

$$P(X=2) = f(2) = \binom{6}{2}\left(\frac{1}{5}\right)^{2}\left(\frac{4}{5}\right)^{6-2} = 15\left(\frac{4^4}{5^6}\right)$$

(ii) Probability for at most one defective product is

$$P(X \leq 1) = P(X=0) + P(X=1)$$

$$= \binom{6}{0}\left(\frac{1}{5}\right)^{0}\left(\frac{4}{5}\right)^{6-0} + \binom{6}{1}\left(\frac{1}{5}\right)^{1}\left(\frac{4}{5}\right)^{6-1}$$

$$= \left(\frac{4}{5}\right)^{6} + (6)\left(\frac{4^5}{5^6}\right) = 2\left(\frac{4}{5}\right)^{5}$$

Probability for at most one defective product is $2\left(\frac{4}{5}\right)^{5}$.

(iii) Probability for at least two defective products is

$$P(X \geq 2) = 1 - P(X<2) = 1 - P(X \leq 1) = 1 - 2\left(\frac{4}{5}\right)^{5}$$

Probability for at least two defective products is $1-2\left(\frac{4}{5}\right)^{5}$.

EXERCISE 11.5

1. Compute $P(X=k)$ for the binomial distribution, $B(n,p)$ where

   (i) $n=6,\ p=\frac{1}{3},\ k=3$ (ii) $n=10,\ p=\frac{1}{5},\ k=4$ (iii) $n=9,\ p=\frac{1}{2},\ k=7$

2. The probability that Mr.Q hits a target at any trial is $\frac{1}{4}$. Suppose he tries at the target 10 times. Find the probability that he hits the target (i) exactly 4 times (ii) at least one time.
3. Using binomial distribution find the mean and variance of X for the following experiments
   (i) A fair coin is tossed 100 times, and X denote the number of heads.
   (ii) A fair die is tossed 240 times, and X denote the number of times that four appeared.
4. The probability that a certain kind of component will survive a electrical test is $\frac{3}{4}$. Find the probability that exactly 3 of the 5 components tested survive.
5. A retailer purchases a certain kind of electronic device from a manufacturer. The manufacturer indicates that the defective rate of the device is 5%.
   The inspector of the retailer randomly picks 10 items from a shipment. What is the probability that there will be (i) at least one defective item (ii) exactly two defective items?
6. If the probability that a fluorescent light has a useful life of at least 600 hours is 0.9, find the probabilities that among 12 such lights
   (i) exactly 10 will have a useful life of at least 600 hours;
   (ii) at least 11 will have a useful life of at least 600 hours;
   (iii) at least 2 will *not* have a useful life of at least 600 hours.
7. The mean and standard deviation of a binomial variate X are respectively 6 and 2.
   Find (i) the probability mass function (ii) $P(X=3)$ (iii) $P(X \geq 2)$.
8. If $X \sim B(n,p)$ such that $4P(X=4)=P(X=2)$ and $n=6$. Find the distribution, mean and standard deviation of X.
9. In a binomial distribution consisting of 5 independent trials, the probability of 1 and 2 successes are 0.4096 and 0.2048 respectively. Find the mean and variance of the random variable.

EXERCISE 11.6

Choose the Correct or the most suitable answer from the given four alternatives:

1. Let X be random variable with probability density function

$$f(x)=\begin{cases}\frac{2}{x^3} & x \geq 1\\ 0 & x<1\end{cases}$$

F1Y9N6

Which of the following statement is correct?

(1) both mean and variance exist (2) mean exists but variance does not exist

(3) both mean and variance do not exist (4) variance exists but Mean does not exist.

2. A rod of length $2l$ is broken into two pieces at random. The probability density function of the shorter of the two pieces is

$$f(x) = \begin{cases} \frac{1}{l} & 0 < x < l \\ 0 & l \le x < 2l \end{cases}$$

The mean and variance of the shorter of the two pieces are respectively

(1) $\frac{l}{2}, \frac{l^2}{3}$ (2) $\frac{l}{2}, \frac{l^2}{6}$ (3) $l, \frac{l^2}{12}$ (4) $\frac{l}{2}, \frac{l^2}{12}$

3. Consider a game where the player tosses a six-sided fair die. If the face that comes up is 6, the player wins ₹ 36, otherwise he loses ₹ k^2, where k is the face that comes up $k = \{1, 2, 3, 4, 5\}$.

The expected amount to win at this game in ₹ is

(1) $\frac{19}{6}$ (2) $-\frac{19}{6}$ (3) $\frac{3}{2}$ (4) $-\frac{3}{2}$

4. A pair of dice numbered 1, 2, 3, 4, 5, 6 of a six-sided die and 1, 2, 3, 4 of a four-sided die is rolled and the sum is determined. Let the random variable X denote this sum. Then the number of elements in the inverse image of 7 is

(1) 1 (2) 2 (3) 3 (4) 4

5. A random variable X has binomial distribution with $n = 25$ and $p = 0.8$ then standard deviation of X is

(1) 6 (2) 4 (3) 3 (4) 2

6. Let X represent the difference between the number of heads and the number of tails obtained when a coin is tossed n times. Then the possible values of X are

(1) $i+2n,\ i = 0,1,2...n$ (2) $2i-n,\ i = 0,1,2...n$ (3) $n-i,\ i = 0,1,2...n$ (4) $2i+2n,\ i = 0,1,2...n$

7. If the function $f(x) = \frac{1}{12}$ for $a < x < b$, represents a probability density function of a continuous random variable X, then which of the following cannot be the value of a and b?

(1) 0 and 12 (2) 5 and 17 (3) 7 and 19 (4) 16 and 24

8. Four buses carrying 160 students from the same school arrive at a football stadium. The buses carry, respectively, 42, 36, 34, and 48 students. One of the students is randomly selected. Let X denote the number of students that were on the bus carrying the randomly selected student. One of the 4 bus drivers is also randomly selected. Let Y denote the number of students on that bus.
Then $E(X)$ and $E(Y)$ respectively are

(1) $50, 40$ (2) $40, 50$ (3) $40.75, 40$ (4) $41, 41$

9. Two coins are to be flipped. The first coin will land on heads with probability 0.6, the second with Probability 0.5. Assume that the results of the flips are independent, and let X equal the total number of heads that result. The value of $E(X)$ is

(1) 0.11 (2) 1.1 (3) 11 (4) 1

10. On a multiple-choice exam with 3 possible destructives for each of the 5 questions, the probability that a student will get 4 or more correct answers just by guessing is

(1) $\frac{11}{243}$ (2) $\frac{3}{8}$ (3) $\frac{1}{243}$ (4) $\frac{5}{243}$

11. If $P(X=0) = 1 - P(X=1)$. If $E(X) = 3\text{Var}(X)$, then $P(X=0)$ is

(1) $\frac{2}{3}$ (2) $\frac{2}{5}$ (3) $\frac{1}{5}$ (4) $\frac{1}{3}$

12. If X is a binomial random variable with expected value 6 and variance 2.4, then $P(X=5)$ is

(1) $\binom{10}{5}\left(\frac{3}{5}\right)^6\left(\frac{2}{5}\right)^4$ (2) $\binom{10}{5}\left(\frac{3}{5}\right)^{10}$

(3) $\binom{10}{5}\left(\frac{3}{5}\right)^4\left(\frac{2}{5}\right)^6$ (4) $\binom{10}{5}\left(\frac{3}{5}\right)^5\left(\frac{2}{5}\right)^5$

13. The random variable X has the probability density function

$$f(x) = \begin{cases} ax+b & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

and $E(X) = \frac{7}{12}$, then a and b are respectively

(1) 1 and $\frac{1}{2}$ (2) $\frac{1}{2}$ and 1 (3) 2 and 1 (4) 1 and 2

14. Suppose that X takes on one of the values 0, 1, and 2. If for some constant k, $P(X=i) = kP(X=i-1)$ for $i = 1,2$ and $P(X=0) = \frac{1}{7}$, then the value of k is

(1) 1 (2) 2 (3) 3 (4) 4

15. Which of the following is a discrete random variable?

I. The number of cars crossing a particular signal in a day.

II. The number of customers in a queue to buy train tickets at a moment.

III. The time taken to complete a telephone call.

(1) I and II (2) II only (3) III only (4) II and III

16. If $f(x) = \begin{cases} 2x & 0 \le x \le a \\ 0 & \text{otherwise} \end{cases}$ is a probability density function of a random variable, then the value of a is

(1) 1 (2) 2 (3) 3 (4) 4

17. The probability mass function of a random variable is defined as:

x	–2	–1	0	1	2
$f(x)$	k	$2k$	$3k$	$4k$	$5k$

Then $E(X)$ is equal to:

(1) $\frac{1}{15}$ (2) $\frac{1}{10}$ (3) $\frac{1}{3}$ (4) $\frac{2}{3}$

18. Let X have a Bernoulli distribution with mean 0.4, then the variance of $(2X-3)$ is

(1) 0.24 b) 0.48 (3) 0.6 (4) 0.96

19. If in 6 trials, X is a binomial variable which follows the relation $9P(X=4) = P(X=2)$, then the probability of success is

(1)0.125 (2) 0.25 (3) 0.375 (4) 0.75

20. A computer salesperson knows from his past experience that he sells computers to one in every twenty customers who enter the showroom. What is the probability that he will sell a computer to exactly two of the next three customers?

(1) $\frac{57}{20^3}$ (2) $\frac{57}{20^2}$ (3) $\frac{19^3}{20^3}$ (4) $\frac{57}{20}$

SUMMARY

- ***A random variable*** *X is a function defined on a sample space S into the real numbers $\mathbb{R}$ such that the inverse image of points or subset or interval of $\mathbb{R}$ is an event in S, for which probability is assigned.*
- A random variable X is defined on a sample space S into the real numbers $\mathbb{R}$ is called discrete random variable if the range of X is countable, that is, it can assume only a finite or countably infinite number of values, where every value in the set S has positive probability with total one.
- If X is a discrete random variable with discrete values $x_1, x_2, x_3, \ldots x_n, \ldots$, then the function denoted by $f(.)$ or $p(.)$ and defined by $f(x_k) = P(X = x_k)$ for $k = 1,2,3,\ldots n,\ldots$ is called the probability mass function of X
- The function $f(x)$ is a probability mass function if

  (i) $f(x_k) \geq 0$ for $k = 1,2,3,\ldots n,\ldots$ and (ii) $\sum_k f(x_k) = 1$
- The **cumulative distribution function** $F(x)$ of a discrete random variable X, taking the values $x_1, x_2, x_3, \ldots$ such that $x_1 < x_2 < x_3 < \cdots$ with probability mass function $f(x_i)$ is

  $$F(x) = P(X \leq x) = \sum_{x_i \leq x} f(x_i), \quad x \in \mathbb{R}$$
- Suppose X is a discrete random variable taking the values $x_1, x_2, x_3, \ldots$ such that $x_1 < x_2 < x_3, \ldots$ and $F(x_i)$ is the distribution function. Then the probability mass function $f(x_i)$ is given by $f(x_i) = F(x_i) - F(x_{i-1})$, $i = 1,2,3, \ldots$
- *Let S be a sample space and let a random variable $X : S \to R$ that takes any value in a set I of $\mathbb{R}$. Then X is called a* ***continuous random variable*** *if $P(X = x) = 0$ for every x in I*
- *A non-negative real valued function $f(x)$ is said to be a* ***probability density function*** *if, for each possible outcome x, $x \in [a,b]$ of a continuous random variable X having the property*

  $$P(a \leq x \leq b) = \int_a^b f(x)\, dx$$

- *Suppose F(x) is the distribution function of a continuous random variable X. Then the probability density function f(x) is given by*

$$f(x) = \frac{dF(x)}{dx} = F'(x), \text{ whenever derivative exists.}$$

- *Suppose X is a random variable with probability mass or density function f(x)* ***The expected value or mean or mathematical expectation of*** *X***,** denoted by $E(x)$ or μ *is*

$$E(X) = \begin{cases} \sum_x xf(x) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} xf(x)\,dx & \text{if } X \text{ is continuous} \end{cases}$$

- *The* ***variance*** *of the random variable X* denoted by $V(X)$ or σ^2 (or σ_x^2) is

$$V(X) = E(X - E(X))^2 = E(X - \mu)^2$$

Properties of Mathematical expectation and variance

(i) $E(aX + b) = aE(X) + b$, where a and b are constants

Corollary 1: $E(aX) = aE(X)$	(when $b = 0$)
Corollary 2: $E(b) = b$	(when $a = 0$)

(ii) $Var(X) = E(X^2) - (E(X))^2$

(iii) $Var(aX + b) = a^2 Var(X)$ where a and b are constants

Corollary 3: $V(aX) = a^2V(X)$	(when $b = 0$)
Corollary 4: $V(b) = 0$	(when $a = 0$)

- Let X be a random variable associated with a Bernoulli trial by defining it as X (success) = 1 and X (failure) = 0, such that

$$f(x) = \begin{cases} p & x = 1 \\ q = 1 - p & x = 0 \end{cases} \quad \text{where } 0 < p < 1$$

- X is called a Bernoulli random variable and $f(x)$ is called the Bernoulli distribution.

- *If X* is a Bernoulli's random variable which follows Bernoulli distribution with parameter p, the mean μ and variance σ^2 are

$$\mu = p \quad \text{and} \quad \sigma^2 = pq$$

- A discrete random variable X is called binomial random variable, if X is the number of successes in n-repeated trials such that

  (i) The n- repeated trials are independent and n is finite

  (ii) Each trial results only two possible outcomes, labelled as 'success' or 'failure'

  (iii) The probability of a success in each trial, denoted as p, remains constant

- The binomial random variable X equals the number of successes with probability p for a success and $q = 1 - p$ for a failure in n-independent trials, has ***a binomial distribution*** denoted by $X \sim B(n, p)$. The probability mass function of X is $f(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, 2, \ldots, n.$

- If X is a binomial random variable which follows binomial distribution with parameters p and n, the mean μ and variance σ^2 are $\mu = np$ and $\sigma^2 = np(1-p)$.

ICT CORNER

https://ggbm.at/dy9kwgbt or Scan the QR Code

Open the Browser, type the URL Link given (or) Scan the QR code. GeoGebra work book named "**12th Standard Mathematics Vol-2**" will open. In the left side of work book there are chapters related to your text book. Click on the chapter named "**Probability Distributions**". You can see several work sheets related to the chapter. Go through all the work sheets.

B262_12_MATHS_EM

Chapter

12

Discrete Mathematics

C2T2H7

"Young man, in mathematics you don't understand things.
You just get used to them".

-John von Neumann

12.1 Introduction

Mathematics can be broadly classified into two categories: **Continuous Mathematics** − It is based upon the results concerning the set of real numbers which is **uncountably infinite**. It is characterized by the fact that between any two real numbers, there is always a set of uncountably infinite numbers. For example, a function in continuous mathematics can be plotted in a smooth curve without break.

John von Neumann
(1903-1957)

Discrete Mathematics − It involves distinct values which are either **finite or countably infinite**; i.e. between any two points, there are finite or countably infinite number of points. For example, if we have a finite set of objects, the function can be defined as a list of ordered pairs having these objects, and can be presented as a complete list of those pairs.

The mathematicians who lived in the latter part of the 19^{th} and early in the 20^{th} centuries developed a new branch of mathematics called **discrete mathematics** consisting of concepts based on either **finite** or **countably infinite** sets like the set of natural numbers. These sets are called discrete sets and the beauty of such sets is that, one can find that a one-to-one correspondence can be defined from these sets onto the set of natural numbers. So, the elements of a discrete set can be arranged as a sequence. This special feature of discrete sets cannot be found in any uncountable set like the set of real numbers where the elements are distributed continuously throughout without any gap.

Everyone is aware of the fact that the application of computers is playing an important role in every walk of our lives. Consequently the **computer science** has become partially a science of clear understanding and concise description of computable discrete sets. Also the modern programming languages are to be designed in such a way that they are suitable for descriptions in a concise manner. This compels the computer scientists to train themselves in learning to formulate algorithms based on the discrete sets.

The main advantage of studying discrete mathematics is that its results serve as very good tools for improving the reasoning and problems solving capabilities. Some of the branches of discrete mathematics are **combinatorics, mathematical logic, boolean algebra, graph theory, coding theory** etc. Some of the topics of discrete mathematics namely **permutations, combinations, and mathematical induction** were already discussed in the previous year. In the present chapter, two topics namely **binary operations** and **mathematical logic** of discrete mathematics are discussed.

Symbols

$\in$	–	belongs to.
$\ni$	–	such that.
$\forall$	–	for every.
$\Rightarrow$	–	implies.
$\exists$	–	there exists

In general, the word 'operation' refers to the process of operating upon either a single or more number of elements at a time. For instance, finding the negative of an element in $\mathbb{Z}$ involves a single element at a time. So it is called an **unary operation.** On the other hand the process of finding the sum of any two elements in $\mathbb{Z}$ involves two elements at a time**.** This kind of operation is called a **binary operation** and in general an operation involving n elements is called an **n-ary operation,** $n \in \mathbb{N}$**.** In this section a detailed discussion of the binary operations is presented**.**

Learning Objectives

Upon completion of this chapter, students will be able to

- define binary operation and examine various properties
- define binary operation on Boolean matrices and verify various properties
- define binary operation on modular classes and examine various properties
- identify simple and compound statements
- define logical connectives and construct truth tables
- identify tautology, contradiction, and contingency
- establish logical equivalence and apply duality principle

12.2 Binary Operations

12.2.1 Definitions

The basic arithmetic operations on $\mathbb{R}$ are **addition** $(+)$**, subtraction** $(-)$, **multiplication** $(\times)$**,** and **division** (÷). Eminent mathematicians of the latter part of 19th century and in 20th century like Abel, Cayley, Cauchy, and others, tried to generalize the properties satisfied by these usual arithmetic operations. To this end they developed new abstract algebraic structures through the **axiomatic approach**. This new branch of algebra dealing with these abstract algebraic structures is known as **abstract algebra.**

To begin with, consider a simple example involving the basic usual arithmetic operations addition and multiplication of any two natural numbers.

$$m+n \in \mathbb{N};\ m\times n \in \mathbb{N},\ \forall m,n \in \mathbb{N} = \{1,2,3,...\}$$

Each of the above two operations yields the following observations:

(1) At a time exactly two elements of $\mathbb{N}$ are processed.

(2) The resulting element (outcome) is also an element of $\mathbb{N}$.

Any such operation defined on a nonempty set is called a **binary operation** or **a binary composition** on the set in abstract algebra.

Definition 12.1

Any operation $*$ defined on a non-empty set S is called a **binary operation** on S if the following conditions are satisfied:

(i) The operation $*$ must be defined for each and every ordered pair $(a,b) \in S \times S$.

(ii) It assigns a **unique** element $a*b$ of S to every ordered pair $(a,b) \in S \times S$.

In other words, any binary operation $*$ on S is a rule that assigns to **each ordered pair** of elements of S a **unique** element of S. Also $*$ can be regarded as a **function (mapping)** with input in the Cartesian product $S \times S$ and the output in S.

$$*: S \times S \to S \;\; ; \;\; *(a,b) = a*b \in S, \text{ where } a*b \text{ is an unique element.}$$

A binary operation defined by $*: S \times S \to S$; $*(a,b) = a*b \in S$ demands that the output $a*b$ must always lie the given set S and not in the complement of it. Then we say that '$*$ is closed on S' or 'S is **closed** with respect to $*$'. This property is known as the **closure property.**

Definition 12.2

Any non-empty set on which one or more binary operations are defined is called an **algebraic structure.**

Another way of defining a binary operation $*$ on S is as follows: $\forall a,b \in S$, $a*b$ is unique and $a*b \in S$.

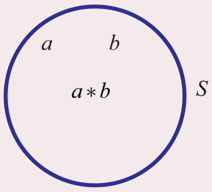


Fig. 12.1

Note

It follows that every binary operation satisfies the closure property.

Note

The operation $*$ is just a symbol which may be $+, \times, -, \div$ matrix addition, matrix multiplication, etc. depending on the set on which it is defined.

For instance, though $+$ and $\times$ are binary on $\mathbb{N}$, $-$ is **not** binary operation on $\mathbb{N}$.

To verify this, consider $(3,4) \in \mathbb{N} \times \mathbb{N}$.

$$*\,(a,b) = -\,(3,4) = 3-4 = -1 \notin \mathbb{N}.$$

Hence $-$ is **not binary operation** on $\mathbb{N}$. So $\mathbb{N}$ is to be extended to $\mathbb{Z}$ in order that $-$ becomes binary operation on $\mathbb{Z}$. Thus $\mathbb{Z}$ is closed with respect to $+, \times$, and $-$. Thus $(\mathbb{Z}, +, \times, -)$ is an algebraic structure.

Observations

The binary operation depends on the set on which it is defined.

(a) The operation – which is **not binary operation** on $\mathbb{N}$ but it is binary on $\mathbb{Z}$. The set $\mathbb{N}$ is extended to include negative numbers. We call the included set $\mathbb{Z}$.

(b) The operation $\div$ on $\mathbb{Z}$ is **not binary operation** on $\mathbb{Z}$. For instance, for $(1,2) \in \mathbb{Z} \times \mathbb{Z}$, $\div\,(1,\ 2) = \frac{1}{2} \notin \mathbb{Z}$. Hence $\mathbb{Z}$ has to be extended further into $\mathbb{Q}$.

(c) It is a known fact that the division by 0 is **not** defined in basic arithmetic. So $\div$ is binary operation on the set $\mathbb{Q} \setminus \{0\}$. Thus $+, \times, -$ are binary operation on $\mathbb{Q}$ and $\div$ is binary operation on $\mathbb{Q} \setminus \{0\}$.

Now the question is regarding the reasons for extending further $\mathbb{Q}$ to $\mathbb{R}$ and then from $\mathbb{R}$ to $\mathbb{C}$. Accordingly, a number system is needed where not only all the basic arithmetic operations $+, -, \times, \div$ but also to include the roots of the equations of the form "$x^2 - 2 = 0$" and "$x^2 + 1 = 0$".

So, in addition to the existing systems, the collection of irrational numbers and imaginary numbers (See Chapter 3) are to be adjoined. Consequently $\mathbb{R}$ and then $\mathbb{C}$ are obtained. The biggest number system $\mathbb{C}$ properly includes all the other number systems $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ as subsets.

Number System / Operation	$\mathbb{N}$	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{Q}\setminus\{0\}$	$\mathbb{R}\setminus\{0\}$	$\mathbb{C}\setminus\{0\}$
$+$	Binary	Binary	Binary	Binary	Binary	Not Binary	Not Binary	Not Binary
$-$	Not Binary	Binary	Binary	Binary	Binary	Not Binary	Not Binary	Not Binary
$\times$	Binary	Binary	Binary	Binary	Binary	Binary	Binary	Binary
$\div$	Not Binary	Not Binary	Not Binary	Not Binary	Not Binary	Binary	Binary	Binary

Table12.1

Example12.1

Examine the binary operation (closure property) of the following operations on the respective sets (if it is not, make it binary):

(i) $a*b = a+3ab-5b^2; \forall a,b \in \mathbb{Z}$ (ii) $a*b = \left(\frac{a-1}{b-1}\right), \forall a,b \neq 1 \in \mathbb{Q}$

Solution

(i) Since $\times$ is binary operation on $\mathbb{Z}$, $a,\ b\in\mathbb{Z} \Rightarrow a\times b = ab \in \mathbb{Z}$ and $b\times b = b^2 \in \mathbb{Z}$ ... (1)

The fact that $+$ is binary operation on $\mathbb{Z}$ and (1) $\Rightarrow 3ab = (ab+ab+ab)\in\mathbb{Z}$ and $5b^2 = (b^2+b^2+b^2+b^2+b^2)\in\mathbb{Z}$. .... (2)

Also $a\in\mathbb{Z}$ and $3ab\in\mathbb{Z}$ implies $a+3ab\in\mathbb{Z}$. ... (3)

(2), (3), the closure property of $-$ on $\mathbb{Z}$ yield $a*b = (a+3ab-5b^2)\in\mathbb{Z}$. Since $a*b$ belongs to $\mathbb{Z}$, * is a binary operation on $\mathbb{Z}$.

(ii) In this problem $a*b$ is in the quotient form. Since the division by 0 is undefined, the denominator $b-1$ must be nonzero.

It is clear that $b-1=0$ if $b=1$. As $1\in\mathbb{Q}$, $*$ is not a binary operation on the whole of $\mathbb{Q}$. However it can be found that by omitting 1 from $\mathbb{Q}$, the output $a*b$ exists in $\mathbb{Q}\setminus\{1\}$. Hence $*$ is a binary operation on $\mathbb{Q}\setminus\{1\}$.

12.2.2 Some more properties of a binary operation

Commutative property

Any binary operation $*$ defined on a nonempty set S is said to satisfy the commutative property, if

$$a*b = b*a \quad \forall a,b\in S.$$

Associative property

Any binary operation $*$ defined on a nonempty set S is said to satisfy the associative property, if

$$a*(b*c)=(a*b)*c \quad \forall a,b,c\in S.$$

Existence of identity property

An element $e\in S$ is said to be the **Identity Element** of S under the binary operation $*$ if for all $a\in S$ we have that $a*e=a$ and $e*a=a$.

Existence of inverse property

If an identity element e exists and if for every $a\in S$, there exists b in S such that $a*b=e$ and $b*a=e$ then $b\in S$ is said to be the **Inverse Element** of a. In such instances, we write $b=a^{-1}$.

Note

a^{-1} is an element of S. It should be read as the inverse of a and not as $\frac{1}{a}$.

Note

(i) The **multiplicative identity** is 1 in $\mathbb{Z}$ and it is the one and only one element with the property $n\cdot 1=1\cdot n=n,\ \forall n\in\mathbb{Z}$.

(ii) The **multiplicative inverse** of any element, say 2 in $\mathbb{Q}$ is $\frac{1}{2}$ and no other nonzero rational number x has the property that $2\cdot x=x\cdot 2=1$.

Note

Whenever a mathematical statement involves 'for every' or ' for all' , it has to be proved for every pair or three elements. It is not easy to prove for every pair or three elements. But these types of definitions may be used to prove the negation of the statement. That is, negation of "for every" or "for all" is "there exists not". So, produce one such pair or three elements to establish the negation of the statement.

The questions of existence and uniqueness of identity and inverse are to be examined. The following theorems prove these results in the more general form.

Theorem 12.1: (Uniqueness of Identity)
In an algebraic structure the identity element (if exists) must be unique.

Proof

Let $(S,*)$ be an algebraic structure. Assume that the identity element of S exists in S.

It is to be proved that the identity element is unique. Suppose that e_1 and e_2 be any two identity elements of S.

First treat e_1 as the identity and e_2 as an arbitrary element of S.

Then by the existence of identity property, $e_2*e_1=e_1*e_2=e_2$. ... (1)

Interchanging the role of e_1 and e_2, $e_1*e_2=e_2*e_1=e_1$. ...(2)

From (1) and (2), $e_1=e_2$. Hence the identity element is unique which completes the proof.

Theorem 12.2 (Uniqueness of Inverse)
In an algebraic structure the inverse of an element (if exists) must be unique.

Proof

Let $(S,*)$ be an algebraic structure and $a\in S$. Assume that the inverse of a exists in S. It is to be proved that the inverse of a is unique. The existence of inverse in S ensures the existence of the identity element e in S.

Let $a \in S$. It is to be proved that the inverse a (if exists) is unique.

Suppose that a has two inverses, say, a_1, a_2.

Treating a_1 as an inverse of a gives $a * a_1 = a_1 * a = e$ ...(1)

Next treating a_2 as the inverse of a gives $a * a_2 = a_2 * a = e$ ...(2)

$$a_1 = a_1 * e = a_1 * (a * a_2) = (a_1 * a) * a_2 = e * a_2 = a_2 \text{ (by (1) and (2)).}$$

So, $a_1 = a_2$. Hence the inverse of a is unique which completes the proof.

Example 12.2

Verify the (i) closure property, (ii) commutative property, (iii) associative property (iv) existence of identity and (v) existence of inverse for the arithmetic operation $+$ on $\mathbb{Z}$.

Solution

(i) $m + n \in \mathbb{Z}, \forall m, n \in \mathbb{Z}$. Hence $+$ is a binary operation on $\mathbb{Z}$.

(ii) Also $m + n = n + m, \forall m, n \in \mathbb{Z}$. So the commutative property is satisfied

(iii) $\forall m, n, p \in \mathbb{Z}, m + (n + p) = (m + n) + p$. Hence the associative property is satisfied.

(iv) $m + e = e + m = m \Rightarrow e = 0$. Thus $\exists\ 0 \in \mathbb{Z} \ni (m + 0) = (0 + m) = m$. Hence the existence of identity is assured.

(v) $m + m' = m' + m = 0 \Rightarrow m' = -m$. Thus $\forall m \in \mathbb{Z}, \exists\ -m \in \mathbb{Z} \ni$
$m + (-m) = (-m) + m = 0$. Hence, the existence of inverse property is also assured. Thus we see that the usual addition $+$ on $\mathbb{Z}$ satisfies all the above five properties.

Note that the **additive identity** is 0 and the **additive inverse** of any integer m is $-m$.

Example 12.3

Verify the (i) closure property, (ii) commutative property, (iii) associative property (iv) existence of identity and (v) existence of inverse for the arithmetic operation $-$ on $\mathbb{Z}$.

Solution

(i) Though $-$ is not binary on $\mathbb{N}$; it is binary on $\mathbb{Z}$. To check the validity of any more properties satisfied by – on $\mathbb{Z}$, it is better to check them for some particular simple values.

(ii) Take $m = 4$, $n = 5$ and $(m - n) = (4 - 5) = -1$ and $(n - m) = (5 - 4) = 1$.
Hence $(m - n) \neq (n - m)$. So the operation $-$ is not commutative on $\mathbb{Z}$.

(iii) In order to check the associative property, let us put $m = 4$, $n = 5$ and $p = 7$ in both $(m - n) - p$ and $m - (n - p)$.

$(m - n) - p = (4 - 5) - 7 = (-1 - 7) = -8$ ...(1)

$m - (n - p) = 4 - (5 - 7) = (4 + 2) = 6$. ...(2)

From (1) and (2), it follows that $(m - n) - p \neq m - (n - p)$.

Hence – is not associative on $\mathbb{Z}$.

(iv) Identity does not exist (why?).

(v) Inverse does not exist (why?).

Example 12.4

Verify the (i) closure property, (ii) commutative property, (iii) associative property (iv) existence of identity and (v) existence of inverse for the arithmetic operation $+$ on $\mathbb{Z}_e$ = the set of all even integers.

Solution

Consider the set of all **even integers** $\mathbb{Z}_e = \{2k \mid k \in \mathbb{Z}\} = \{..., -6, -4, -2, 0, 2, 4, 6, ...\}$.

Let us verify the properties satisfied by $+$ on $\mathbb{Z}_e$.

(i) The sum of any two even integers is also an even integer.

Because $x, y \in \mathbb{Z}_e \Rightarrow x = 2m$ and $y = 2n$, $m, n \in \mathbb{Z}$.

So $(x+y) = 2m + 2n = 2(m+n) \in \mathbb{Z}_e$. Hence $+$ is a binary operation on $\mathbb{Z}_e$.

(ii) $\forall x, y \in \mathbb{Z}_e, (x+y) = 2(m+n) = 2(n+m) = (2n+2m) = (y+x)$.

So $+$ has commutative property.

(iii) Similarly it can be seen that $\forall x, y, z \in \mathbb{Z}_e$, $(x+y)+z = x+(y+z)$.

Hence the associative property is true.

(iv) Now take $x = 2k$, then $2k + e = e + 2k = 2k \Rightarrow e = 0$.

Thus $\forall x \in \mathbb{Z}_e$, $\exists 0 \in \mathbb{Z}_e \ni x + 0 = 0 + x = x$.

So, 0 is the identity element.

(v) Taking $x = 2k$ and x' as its inverse, we have $2k + x' = 0 = x' + 2k \Rightarrow x' = -2k$. *i.e.,* $x' = -x$.

Thus $\forall x \in \mathbb{Z}_e, \exists -x \in \mathbb{Z}_e \ni x + (-x) = (-x) + x = 0$

Hence $-x$ is the inverse of $x \in \mathbb{Z}_e$.

Example 12.5

Verify the (i) closure property, (ii) commutative property, (iii) associative property (iv) existence of identity and (v) existence of inverse for the arithmetic operation $+$ on $\mathbb{Z}_o$ = the set of all odd integers.

Solution

Consider the set $\mathbb{Z}_o$ of all **odd integers** $\mathbb{Z}_o = \{2k+1 : k \in \mathbb{Z}\} = \{..., -5, -3, -1, 1, 3, 5, ...\}$. $+$ is **not a binary operation** on $\mathbb{Z}_o$ because when $x = 2m+1, y = 2n+1, x+y = 2(m+n)+2$ is even for all m and n. For instance, consider the two odd numbers $3, 7 \in \mathbb{Z}_o$. Their sum $3 + 7 = 10$ is an even number. In general, if $x, y \in \mathbb{Z}_0$, then $(x+y) \notin \mathbb{Z}_0$. Other properties need not be checked as it is not a binary operation.

Example 12.6

Verify (i) closure property (ii) commutative property, and (iii) associative property of the following operation on the given set.

$(a * b) = a^b; \forall a, b \in \mathbb{N}$ (exponentiation property)

Solution

(i) It is true that $a * b = a^b \in \mathbb{N}; \forall a, b \in \mathbb{N}$. So $*$ is a **binary operation** on $\mathbb{N}$.

(ii) $a * b = a^b$ and $b * a = b^a$. Put, $a = 2$ and $b = 3$. Then $a * b = 2^3 = 8$ but $b * a = 3^2 = 9$

So $a * b$ need not be equal to $b * a$. Hence $*$ **does not have commutative property**.

(iii) Next consider $a * (b * c) = a * (b^c) = a^{(b^c)}$. Take $a = 2$, $b = 3$ and $c = 4$.

Then $a * (b * c) = 2 * (3 * 4) = 2^{3^4} = 2^{81}$

But $(a * b) * c = (a^b) * c = (a^b)^c = a^{(bc)} = a^{bc} = 2^{12}$

Hence $a * (b * c) \neq (a * b) * c$. So $*$ **does not have associative property** on $\mathbb{N}$.

Note: This binary operation has no identity and no inverse. (Justify).

Example 12.7

Verify (i) closure property, (ii) commutative property, (iii) associative property, (iv) existence of identity, and (v) existence of inverse for following operation on the given set.

$m * n = m + n - mn;\ m, n \in \mathbb{Z}$

Solution

(i) The output $m + n - mn$ is clearly an integer and hence $*$ is a **binary operation** on $\mathbb{Z}$.

(ii) $m * n = m + n - mn = n + m - nm = n * m,\ \forall m, n \in \mathbb{Z}$. So $*$ has **commutative property.**

(iii) Consider $(m * n) * p = (m + n - mn) * p = (m + n - mn) + p - (m + n - mn)\, p$

$$= m + n + p - mn - mp - np + mnp \quad \ldots (1)$$

Similarly $m * (n * p) = m * (n + p - np) = m + (n + p - np) - m(n + p - np)$

$$= m + n + p - np - mn - mp + mnp \quad \ldots (2)$$

From (1) and (2), we see that $m * (n * p) = (m * n) * p$. Hence $*$ has **associative property.**

(iv) An integer e is to be found such that

$m * e = e * m = m,\ \forall m \in \mathbb{Z} \Rightarrow m + e - m\,e = m$

$\Rightarrow e\,(1 - m) = 0 \Rightarrow e = 0$ or $m = 1$. But m is an arbitrary integer and hence need not be equal to 1. So the only possibility is $e = 0$. Also $m * 0 = 0 * m = m,\ \forall m \in \mathbb{Z}$. Hence 0 is the identity element and hence the **existence of identity** is assured.

(v) An element $m' \in \mathbb{Z}$ is to be found such that $m * m' = m' * m = e = 0,\ \forall m \in \mathbb{Z}$.

$m * m' = 0 \Rightarrow m + m' - m\,m' = 0 \Rightarrow m' = \dfrac{m}{m-1}$. When $m=1,\ m'$ is not defined.

When $m = 2,\ m'$ is an integer. But except $m=2,\ m'$ need not be an integer for all values of m. Hence **inverse does not exist in** $\mathbb{Z}$.

12.2.3 Some binary operations on Boolean Matrices

Definition 12.3

A **Boolean Matrix** is a real matrix whose entries are either 0 or 1.

Note that the boolean entries 0 and 1 can be defined in several ways. In electrical switch to describe "on and off", in graph theory, the "adjacency matrix" etc , the boolean entries 0 and 1 are used. We consider the same type of Boolean matrices in our discussion.

The following two kinds of operations on the collection of all boolean matrices are defined.

Let $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$ be any two boolean matrices of the **same type**. Then their **join** $\vee$ and **meet** $\wedge$ are defined as follows:

Definition 12.4: Join of A and B

$$A \vee B = \left[a_{ij}\right] \vee \left[b_{ij}\right] = \left[a_{ij} \vee b_{ij}\right] = \left[c_{ij}\right]$$

$$\text{where } c_{ij} = \begin{cases} 1, & \text{if either } a_{ij} = 1 \text{ or } b_{ij} = 1 \\ 0, & \text{if both } a_{ij} = 0 \text{ and } b_{ij} = 0 \end{cases}$$

Definition 12.5: Meet of A and B

$$A \wedge B = \left[a_{ij}\right] \wedge \left[b_{ij}\right] = \left[a_{ij} \wedge b_{ij}\right] = \left[c_{ij}\right] \text{where } c_{ij} = \begin{cases} 1, \text{ if both } a_{ij} = 1 \text{ and } b_{ij} = 1 \\ 0, \text{ if either } a_{ij} = 0 \text{ or } b_{ij} = 0. \end{cases}$$

It is clear that $(a \vee b) = \max\{a, b\}$; $(a \wedge b) = \min\{a, b\}$, $a, b \in \{0, 1\}$.

Example 12.8

Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ be any two boolean matrices of the same type. Find $A \vee B$ and $A \wedge B$.

Solution

$$\text{Then } A \vee B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 \vee 1 & 1 \vee 1 \\ 1 \vee 0 & 1 \vee 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$A \wedge B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \wedge \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 \wedge 1 & 1 \wedge 1 \\ 1 \wedge 0 & 1 \wedge 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Properties satisfied by join and meet

Let $\mathbb{B}$ be the set of all boolean matrices of the same type. We only state the properties of meet and join.

Closure property

$A, B \in \mathbb{B}$, $A \vee B = [a_{ij}] \vee [b_{ij}] = [a_{ij} \vee b_{ij}] \in \mathbb{B}$. (Because, $(a_{ij} \vee b_{ij})$ is either 0 or 1 $\forall i, j$. $\vee$ is a binary operation on $\mathbb{B}$.

Associative property

$A \vee (B \vee C) = (A \vee B) \vee C$, $\forall A, B, C \in \mathbb{B}$. $\vee$ is associative.

Existence of identity property

$\forall A \in \mathbb{B}$, $\exists$ the null matrix $0 \in \mathbb{B} \ni A \vee 0 = 0 \vee A = A$. The identity element for $\vee$ is the null matrix.

Existence of inverse property

For any matrix $A \in \mathbb{B}$, it is impossible to find a matrix

$B \in \mathbb{B} \ni A \vee B = B \vee A = 0$. So the inverse does not exist.

Similarly, it can be verified that the operation meet $\wedge$ satisfies (i) closure property (ii) commutative property (iii) associative property (iv) the matrix $U = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ exists as the identity in $\mathbb{B}$ and (v) the existence of inverse is not assured.

12.2.4 Modular Arithmetic

Having discussed the properties of operations like basic usual arithmetic operations, matrix addition and multiplication, join and meet of boolean matrices, one more new operation called the **Modular Arithmetic** is discussed in this section**.** The modular arithmetic refers to the process of **dividing** some number a by a positive integer n (> 1), called modulus, and then equating a with the remainder b modulo n and it is written as $a \equiv b \pmod{n}$, read as 'a is congruent to b modulo n'.

Here $a \equiv b \pmod{n}$ means $a-b = n \cdot k$ for some integer k and b is the **least non-negative integer** when a is divided by n.

For instance, $25 \equiv 4 \pmod{7}, -20 \equiv -2 \pmod{3} \equiv 1 \pmod{3}$ and $15 \equiv 0 \pmod{5}$, etc. Further the set of integers when divided by n, leaves the remainder $0, 1, 2, \ldots, n-1$. In the case of $\mathbb{Z}_5$,

$$\begin{aligned}
[0] &= \{\ldots, -15, -10, -5, 0, 5, 10, 15, \ldots\} \\
[1] &= \{\ldots, -14, -9, -4, 1, 6, 11, \ldots\} \\
[2] &= \{\ldots, -13, -8, -3, 2, 7, 12, \ldots] \\
[3] &= \{\ldots, -12, -7, -2, 3, 8, 13, \ldots\} \\
[4] &= \{\ldots, -11, -6, -1, 4, 9, 14, \ldots\}.
\end{aligned}$$

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We write this as $\mathbb{Z}_5 = \{[0], [1], [2], [3], [4]\}$. In each class, any two numbers are congruent modulo 5.

Before 2007, modular arithmetic is used in 10-digit ISBN (International Standard Book Number) numbering system. For instance, the last digit is for parity check. It is from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, X\}$. In ISBN number, 81-7808-755-**3**, the last digit **3** is obtained as

$$1*8+2*1+3*7+4*8+5*0+6*8+7*7+8*5+9*5=8+2+21+32+0+48+49+40+45=245 \equiv 3 \pmod{11}.$$

Alternatively, the weighted sum is calculated in the reverse manner

9*8+8*1+7*7+6*8+5*0+4*8+3*7+2*5+1*5=245 = **3** (mod 11).

In both ways, we get the same check number 3.

After 2007, 13-digit ISBN numbering has been followed. The first 12 digits (from left to right) are multiplied by the weights 3,1,3,1,…. starting from right to left. Then the weighted sum is calculated. The higher multiple of 10 is taken. Then the difference is calculated. Then its additive inverse modulo 10 is the thirteenth digit.

For instance, consider the ISBN Number: 978-81-931995-6-**5.**Take 12 digits from left to right.

9	7	8	8	1	9	3	1	9	9	5	6
1	3	1	3	1	3	1	3	1	3	1	3
9	21	8	24	1	27	3	3	9	27	5	18

The total of last row is 155. The nearest (higher) integer in multiples of 10 is 160. The difference 160-155=5. The additive inverse modulo 10 is **5** which is 13-th digit in the ISBN number.

Two new operations namely **addition modulo** $n(+_n)$ and **multiplication modulo** $n(\times_n)$ are defined on the set $\mathbb{Z}_n$ of all non-negative integers less than n under modulo arithmetic.

Definition 12.6

(i) The addition modulo n is defined as follows.

Let $a, b \in \mathbb{Z}_n$. Then

$a +_n b =$ the remainder of $a+b$ on division by n.

(ii) The multiplication modulo n is defined as follows.

Let $a, b \in \mathbb{Z}_n$. Then

$a \times_n b$ the remainder of $a \times b$ on division by n

Example 12.9

Verify (i) closure property, (ii) commutative property, (iii) associative property, (iv) existence of identity, and (v) existence of inverse for the operation $+_5$ on $\mathbb{Z}_5$ using table corresponding to addition modulo 5.

Solution

It is known that $\mathbb{Z}_5 = \{[0],\ [1],\ [2],\ [3],\ [4]\}$. The table corresponding to addition modulo 5 is as follows: We take reminders $\{0,1,2,3,4\}$ to represent the classes $\{[0],[1],[2],[3],[4]\}$.

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Table12.2

(i) Since each box in the table is filled by **exactly one element** of $\mathbb{Z}_5$, the output $a +_5 b$ is unique and hence $+_5$ is a **binary operation**.

(ii) The entries are **symmetrically** placed with respect to the **main diagonal**. So $+_5$ has **commutative property**.

(iii) The table cannot be used directly for the verification of the associative property. So it is to be verified as usual.

For instance, $(2 +_5 3) +_5 4 = 0 +_5 4 = 4$ (mod 5)

and $2 +_5 (3 +_5 4) = 2 +_5 2 = 4 (\text{mod}\, 5)$.

Hence $(2 +_5 3) +_5 4 = 2 +_5 (3 +_5 4)$.

Proceeding like this one can verify this for all possible triples and ultimately it can be shown that $+_5$ is associative.

(iv) The row headed by 0 and the column headed by 0 are identical. Hence the identity element is 0.

(v) The existence of inverse is guaranteed provided the identity 0 exists in each row and each column. From Table12.2, it is clear that this property is true in this case. The method of finding the inverse of any one of the elements of $\mathbb{Z}_5$, say 2 is outlined below.

First find the position of the identity element 0 in the III row headed by 2. Move horizontally along the III row and after reaching 0, move vertically above 0 in the IV column, because 0 is in the III row and IV column. The element reached at the topmost position of IV column is 3. This element 3 is nothing but the inverse of 2, because, $2 +_5 3 = 0 (\text{mod}\, 5)$. In this way, the inverse of each and every element of $\mathbb{Z}_5$ can be obtained. Note that the inverse of 0 is 0,that of 1 is 4, that of 2 is 3, that of 3 is 2 , and, that of 4 is 1.

Example 12.10

Verify (i) closure property, (ii) commutative property, (iii) associative property, (iv) existence of identity, and (v) existence of inverse for the operation $\times_{11}$ on a subset $A = \{1,3,4,5,9\}$ of the set of remainders $\{0,1,2,3,4,5,6,7,8,9,10\}$.

Solution

The table for the operation $\times_{11}$ is as follows.

$\times_{11}$	1	3	4	5	9
1	1	3	4	5	9
3	3	9	1	4	5
4	4	1	5	9	3
5	5	4	9	3	1
9	9	5	3	1	4

Table12.3

Following the same kind of procedure as explained in the previous example, a brief outline of the process of verification of the properties of $\times_{11}$ on A is given below.

(i) Since each box has an unique element of A, $\times_{11}$ is a **binary operation** on A.

(ii) The entries are symmetrical about the main diagonal. Hence $\times_{11}$ has **commutative property.**

(iii) As usual, the **associative property** can be seen to be true.

(iv) The entries of both the row and column headed by the element 1 are identical. Hence 1 is the **identity** element.

(v) Since the identity 1 exists in each row and each column, the **existence of inverse** property is assured for $\times_{11}$. The inverse of 1 is 1, that of 3 is 4, that of 4 is 3, 5 is 9 , and, that of 9 is 5.

EXERCISE 12.1

1. Determine whether $*$ is a binary operation on the sets given below.

   (i) $a*b=a.|b|$ on $\mathbb{R}$ (ii) $a*b=\min(a,b)$ on $A=\{1,2,3,4,5\}$

   (iii) $(a*b)=a\sqrt{b}$ is binary on $\mathbb{R}$.

2. On $\mathbb{Z}$, define $*$ by $(m*n)=m^n+n^m : \forall m,n\in\mathbb{Z}$. Is $*$ binary on $\mathbb{Z}$?

3. Let $*$ be defined on $\mathbb{R}$ by $(a*b)=a+b+ab-7$. Is $*$ binary on $\mathbb{R}$? If so, find $3*\left(\frac{-7}{15}\right)$.

4. Let $A=\{a+\sqrt{5}b : a,b\in\mathbb{Z}\}$. Check whether the usual multiplication is a binary operation on A.

5. (i) Define an operation $*$ on $\mathbb{Q}$ as follows: $a*b=\left(\frac{a+b}{2}\right)$; $a,b\in\mathbb{Q}$. Examine the closure, commutative, and associative properties satisfied by $*$ on $\mathbb{Q}$.

   (ii) Define an operation $*$ on $\mathbb{Q}$ as follows: $a*b=\left(\frac{a+b}{2}\right)$; $a,b\in\mathbb{Q}$. Examine the existence of identity and the existence of inverse for the operation $*$ on $\mathbb{Q}$.

6. Fill in the following table so that the binary operation $*$ on $A=\{a,b,c\}$ is commutative.

$*$	a	b	c
a	b		
b	c	b	a
c	a		c

7. Consider the binary operation $*$ defined on the set $A=\{a,b,c,d\}$ by the following table:

$*$	a	b	c	d
a	a	c	b	d
b	d	a	b	c
c	c	d	a	a
d	d	b	a	c

Is it commutative and associative?

8. Let $A=\begin{pmatrix}1&0&1&0\\0&1&0&1\\1&0&0&1\end{pmatrix}$, $B=\begin{pmatrix}0&1&0&1\\1&0&1&0\\1&0&0&1\end{pmatrix}$, $C=\begin{pmatrix}1&1&0&1\\0&1&1&0\\1&1&1&1\end{pmatrix}$ be any three boolean matrices of the same type. Find (i) $A\vee B$ (ii) $A\wedge B$ (iii) $(A\vee B)\wedge C$ (iv) $(A\wedge B)\vee C$.

9. (i) Let $M=\left\{\begin{pmatrix}x&x\\x&x\end{pmatrix}: x\in R-\{0\}\right\}$ and let $*$ be the matrix multiplication. Determine whether M is closed under $*$. If so, examine the commutative and associative properties satisfied by $*$ on M.

(ii) Let $M=\left\{\begin{pmatrix}x&x\\x&x\end{pmatrix}: x\in R-\{0\}\right\}$ and let $*$ be the matrix multiplication. Determine whether M is closed under $*$. If so, examine the existence of identity, existence of inverse properties for the operation $*$ on M.

10. (i) Let A be $\mathbb{Q}\setminus\{1\}$. Define $*$ on A by $x*y=x+y-xy$. Is $*$ binary on A? If so, examine the commutative and associative properties satisfied by $*$ on A.

(ii) Let A be $\mathbb{Q}\setminus\{1\}$. Define $*$ on A by $x*y=x+y-xy$. Is $*$ binary on A? If so, examine the existence of identity, existence of inverse properties for the operation $*$ on A.

12.3 Mathematical Logic

George Boole
(1815-1864)

George Boole was a self-taught English Mathematician, Philosopher and Logician. His results on **Boolean Algebra** involving the binary numbers play an important role in various fields, particularly more in computer applications. He introduced the idea of Symbolic Logic and contributed a lot of results to the fast development of Mathematical Logic.

The reputed Greek philosopher Aristotle (384-322BC(BCE)) wrote the first book on logic. The famous German philosopher and mathematician Gottfried Leibnitz of 17^{th}century framed the idea of using symbols in Logic. Later this idea was realized by George Boole and Augustus de Morgan in 19^{th} century. George Boole established the fact that logic is very much related to mathematics by linking logic, symbols, and algebra together. Mathematical Logic was developed in the late $19^{\text{th}}$and early 20^{th}centuries.

In 1930 the researchers noticed (Neumann's statement in his death bed: *0 and 1 are going to rule the world*) that the binary numbers 0 and 1 could be used to analyze electrical circuits and thus used to design electronic computers. Today digital computers and electronic circuits are designed to implement this binary arithmetic. We study Mathematical Logic as the language and deductive system of Mathematics and Computer Science.

Generally Logic is the study of valid reasoning. But mathematical logic allows us to represent knowledge in a precise mathematical way and it also allows us to make valid inferences using a set of precise rules. It is regarded as a powerful tool for computer science because it is mainly used to verify the correctness of programs.

12.3.1 Statement and its truth value

The simplest part of Mathematical Logic is the **Propositional Logic** and its building blocks are statements or propositions. Mostly communication needs the use of language through which we impart our ideas. They are in the form of sentences.

There are various types of sentences like

(1) Declarative (Assertive type)

(2) Imperative (A command or a request type)

(3) Exclamatory (Emotions, excitement type)

(4) Interrogative (Question type)

(5) Open type

L6Y1L1

Definition 12.7

Any **declarative sentence** is called a **statement** or a **proposition** which is either **true** or **false** but not both.

Any **imperative sentence** such as exclamatory, command and any **interrogative sentence** cannot be a proposition.

The **truth value** of a statement refers to the truth or the falsity of that particular statement. The truth value of a true statement is **true** and it is denoted by T or 1**.** The truth value of a false statement is **false** and it is denoted by F or 0**.**

An **open sentence** is a sentence whose truth can vary according to some conditions, which are not stated in the sentence. For instance, (i) $x \times 7 = 35$ is an open sentence whose truth value depends on value of x. That is, if $x = 5$, it is true and if $x \neq 5$, it is false. (ii) *He is a bad person*. This is an open sentence. Opinion varies from individual to individual.

Example 12.11

Identify the valid statements from the following sentences.

Solution:

(1) Mount Everest is the highest mountain of the world.

(2) $3+4=8$.

(3) $7+5>10$.

(4) Give me that book.

(5) $(10-x)=7$.

(6) How beautiful this flower is!

(7) Where are you going?

(8) Wish you all success.

(9) This is the beginning of the end.

The truth value of the sentences (1) and (3) are T, while that of (2) is F**.** Hence they are statements. The sentence (5) is true for $x=3$ and false for $x \neq 3$ and hence it may be true or false but not both. So it is also a statement.

The sentences (4), (6), (7), (8) are **not** statements, because (4) is a command, (6) is an exclamatory, (7) is a question while (8) is a sentence expressing one's wishes and (9) is a paradox.

12.3.2 Compound Statements, Logical Connectives, and Truth Tables

Definition 12.8: (Simple and Compound Statements)

Any sentence which cannot be split further into two or more statements is called an **atomic statement** or **a simple statement.** If a statement is the combination of two or more simple statements, then it is called a **compound statement** or a **molecular statement**. Hence it is clear that any statement can be either a simple statement or a compound statement.

Example for simple statements

The sentences (1), (2), (3) given in example 12.11 are simple statements.

Example for Compond statements

Consider the statement, "1 is not a prime number and Ooty is in Kerala".

Note that the above statement is actually a combination of the following two simple statements:

p : 1 is not a prime number.

q : Ooty is in Kerala.

Hence the given statement is not a simple statement. It is a compound statement.

From the above discussions, it follows that any simple statement takes the value either T or F. So it can be treated as a variable and this variable is known as **statement variable** or **propositional variable**. The propositional variables are usually denoted by p, q, r, ...

Definition 12.9 : (Logical Connectives)

To connect two or more simple sentences, we use the words or a group of words such as "and", "or", "if-then", "if and only if", and "not". These connecting words are known as **logical connectives.**

In order to construct a compound statement from simple statements, some connectives are used**.** Some basic logical connectives are **negation (not), conjunction (and)** and **disjunction(or).**

Definition 12.10

A **statement formula** is an expression involving one or more statements connected by some logical connectives.

Definition 12.11: (Truth Table)

A table showing the relationship between truth values of simple statements and the truth values of compound statements formed by using these simple statements is called **truth table**.

Definition12.12

(i) Let p be a simple statement. Then the **negation** of p is a statement whose truth value is opposite to that of p. It is denoted by $\neg p$, read as **not** p. The truth value of $\neg p$ is T, if p is F, otherwise it is F.

(ii) Let p and q be any two simple statements. The **conjunction** of p and q is obtained by connecting p and q by the word **and**. It is denoted by $p \wedge q$, read as 'p conjunction q' or 'p hat q'. The truth value of $p \wedge q$ is T, whenever both p and q are T and it is F otherwise.

(iii) The **disjunction** of any two simple statements p and q is the compound statement obtained by connecting p and q by the word 'or'. It is denoted by $p \vee q$, read as 'p disjunction q' or 'p cup q'. The truth value of $p \vee q$ is F, whenever both p and q are F and it is T otherwise.

Logical Connectives and their Truth Tables

(1) Truth Table for NOT [$\neg$] (Negation)

Truth Table for $\neg p$

p	$\neg p$
T	F
F	T

Table 12.4

(2) Truth table for AND [$\wedge$] (Conjunction)

Truth Table for $p \wedge q$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Table 12.5

(3) The truth tables for OR [$\vee$] (Disjunction)

Truth Table for $p \vee q$

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Table 12.6

Example 12.12

Write the statements in words corresponding to $\neg p$, $p \wedge q$, $p \vee q$ and $q \vee \neg p$, where p is 'It is cold' and q is 'It is raining.'

Solution

(1) $\neg p$: It is not cold.

(2) $p \wedge q$: It is cold and raining.

(3) $p \vee q$: It is cold or raining.

(4) $q \vee \neg p$: It is raining or it is not cold

Observe that the statement formula $\neg p$ has only 1 variable p and its truth table has $2 = (2^1)$ rows. Each of the statement formulae $p \wedge q$ and $p \vee q$ has two variables p and q. The truth table corresponding to each of them has $4 = (2^2)$ rows. In general, it follows that if a statement formula involves n variables, then its truth table will contain 2^n rows.

Example 12.13

How many rows are needed for following statement formulae?

(i) $p \vee \neg t \wedge (p \vee \neg s)$ (ii) $((p \wedge q) \vee (\neg r \vee \neg s)) \wedge (\neg t \wedge v)$

Solution

(i) $(p \vee \neg t) \wedge (p \vee \neg s)$ contains 3 variables p, s ,and t. Hence the corresponding truth table will contain $2^3 = 8$ rows.

(ii) $((p \wedge q) \vee (\neg r \vee \neg s)) \wedge (\neg t \wedge v)$ contains 6 variables p, q, r, s, t, and v. Hence the corresponding truth table will contain $2^6 = 64$ rows.

Conditional Statement

Definition 12.13

The conditional statement of any two statements p and q is the statement, "If p, then q" and it is denoted by $p \rightarrow q$. Here p is called the **hypothesis** or **antecedent** and q is called the **conclusion** or **consequence**. $p \rightarrow q$ is false only if p is true and q is false. Otherwise it is true.

Truth table for $p \rightarrow q$

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Table 12.7

Example 12.14

Consider $p \rightarrow q$: If today is Monday, then 4 + 4 = 8.

Here the component statements p and q are given by,

p: Today is Monday; q: 4 + 4 = 8.

The truth value of $p \rightarrow q$ is T because the conclusion q is T.

An important point is that $p \rightarrow q$ should not be treated by actually considering the meanings of p and q in English. Also it is not necessary that p should be related to q at all.

Consequences

From the conditional statement $p \to q$, three more conditional statements are derived. They are listed below.

(i) **Converse statement** $q \to p$.

(ii) **Inverse statement** $\neg p \to \neg q$.

(iii) **Contrapositive statement** $\neg q \to \neg p$.

Example 12.15

Write down the (i) conditional statement (ii) converse statement (iii) inverse statement, and (iv) contrapositive statement for the two statements p and q given below.

p : The number of primes is infinite. q: Ooty is in Kerala.

Solution

Then the four types of conditional statements corresponding to p and q are respectively listed below.

(i) $p \to q$: (conditional statement) "**If** the number of primes is infinite **then** Ooty is in Kerala".

(ii) $q \to p$: (converse statement) "**If** Ooty is in Kerala **then** the number of primes is infinite"

(iii) $\neg p \to \neg q$ (inverse statement) "**If** the number of primes is **not** infinite **then** Ooty is **not** in Kerala".

(iv) $\neg q \to \neg p$ (contrapositive statement) "**If** Ooty is **not** in Kerala **then** the number of primes is **not** infinite".

Bi-conditional Statement

Definition 12.14

The **bi-conditional statement** of any two statements p and q is the statement " p if and only if q " and is denoted by $p \leftrightarrow q$. Its truth value is T , whenever both p and q have the same truth values, otherwise it is false.

Truth table for $p \leftrightarrow q$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Table 12.8

T7D1C3

Exclusive OR (EOR)[$\bar{\vee}$]

Definition 12.15

Let p and q be any two statements. Then p **EOR** q is such a compound statement that its truth value is decided by either p or q but **not both**. It is denoted by $p \bar{\vee} q$. The truth value of $p \bar{\vee} q$ is T whenever either p or q is T, otherwise it is F. The truth table of $p \bar{\vee} q$ is given below.

Truth Table for $p \veebar q$

p	q	$p \veebar q$
T	T	F
T	F	T
F	T	T
F	F	F

Table 12.9

Example 12.16

Construct the truth table for $(p \veebar q) \wedge (p \veebar \neg q)$.

p	q	$\neg q$	$r : (p \veebar q)$	$s : (p \veebar \neg q)$	$r \wedge s$
T	T	F	F	T	F
T	F	T	T	F	F
F	T	F	T	F	F
F	F	T	F	T	F

Table 12.10

Also the above result can be proved without using truth tables. This proof will be provided after studying the logical equivalence.

12.3.3 Tautology, Contradiction, and Contingency

Definition 12.16

A statement is said to be a **tautology** if its truth value is always T irrespective of the truth values of its component statements. It is denoted by $\mathbb{T}$.

Definition 12.17

A statement is said to be a **contradiction** if its truth value is always F irrespective of the truth values of its component statements. It is denoted by $\mathbb{F}$.

Definition 12.18

A statement which is neither a tautology nor a contradiction is called **contingency**

Observations

1. For a tautology, all the entries in the column corresponding to the statement formula will contain T.
2. For a contradiction, all the entries in the column corresponding to the statement formula will contain F.
3. The negation of a tautology is a contradiction and the negation of a contradiction is a tautology.
4. The disjunction of a statement with its negation is a tautology and the conjunction of a statement with its negation is a contradiction. That is $p \vee \neg p$ is a **tautology** and $p \wedge \neg p$ is a **contradiction**. This can be easily seen by constructing their truth tables as given below.

Example for tautology

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

Table 12.11

Since the last column of $p \vee \neg p$ contains only T, $p \vee \neg p$ is a tautology.

Example for contradiction

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

Table 12.12

Since the last column contains only F, $p \wedge \neg p$ is a contradiction.

Note

All the entries in the last column of Table 12.10 are F and hence $(p \veebar q) \wedge (p \veebar \neg q)$ is a contradiction.

Example for contingency

p	q	$p \leftrightarrow q$	$\neg q$	$p \to \neg q$	$\neg(p \to \neg q)$	$(p \leftrightarrow q) \wedge \neg(p \to \neg q)$
T	T	T	F	F	T	T
T	F	F	T	T	F	F
F	T	F	F	T	F	F
F	F	T	T	T	F	F

Table 12.13

In the above truth table, the entries in the last column are a combination of T and F. The given statement is neither a tautology nor a contradiction. It is a contingency.

12.3.4 Duality

Definition 12.19

The **dual** of a statement formula is obtained by replacing $\vee$ by $\wedge$, $\wedge$ by $\vee$, T by F, F by T. A dual is obtained by replacing $\mathbb{T}$ **(tautology)** by $\mathbb{F}$ **(contradiction)**, and, $\mathbb{F}$ by $\mathbb{T}$.

Remarks

(1) The symbol $\neg$ is not changed while finding the dual.

(2) Dual of a dual is the statement itself.

(3) The special statements $\mathbb{T}$ (tautology) and $\mathbb{F}$ (contradiction) are duals of each other.

(4) T is changed to F and vice-versa.

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Principle of Duality

If a compound statement S_1 contains only $\neg$, $\wedge$, and $\vee$ and statement S_2 arises from S_1 by replacing $\wedge$ by $\vee$, and, $\vee$ by $\wedge$ then S_1 is a tautology if and only if S_2 is a contradiction.

For example

(i) The dual of $(p \vee q) \wedge (r \wedge s) \vee \mathbb{F}$ is $(p \wedge q) \vee (r \vee s) \wedge \mathbb{T}$.

(ii) The dual of $p \wedge [\neg q \vee (p \wedge q) \vee \neg r]$ is $p \vee [\neg q \wedge (p \vee q) \wedge \neg r]$.

12.3.5 Logical Equivalence

Definition 12.20

Any two compound statements A and B are said to be **logically equivalent** or simply **equivalent** if the columns corresponding to A and B in the truth table have **identical truth values**. The logical equivalence of the statements A and B is denoted by $A \equiv B$ or $A \Leftrightarrow B$.

From the definition, it is clear that, if A and B are logically equivalent, then $A \Leftrightarrow B$ must be a **tautology**.

Some Laws of Equivalence

1. Idempotent Laws

(i) $p \vee p \equiv p$ (ii) $p \wedge p \equiv p$.

Proof

p	p	$p \vee p$	$p \wedge p$
T	T	T	T
F	F	F	F

Table 12.14

In the above truth table for both p, $p \vee p$ and $p \wedge p$ have the same truth values. Hence $p \vee p \equiv p$ and $p \wedge p \equiv p$.

2. Commutative Laws

(i) $p \vee q \equiv q \vee p$ (ii) $p \wedge q \equiv q \wedge p$.

Proof (i)

p	q	$p \vee q$	$q \vee p$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

Table 12.15

The columns corresponding to $p \vee q$ and $q \vee p$ are identical. Hence $p \vee q \equiv q \vee p$.

Similarly (ii) $p \wedge q \equiv q \wedge p$ can be proved.

3. Associative Laws

(i) $p \vee (q \vee r) \equiv (p \vee q) \vee r$ (ii) $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$.

Proof

The truth table required for proving the associative law is given below.

p	q	r	$p \vee q$	$q \vee r$	$(p \vee q) \vee r$	$p \vee (q \vee r)$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	T	T	T
T	F	F	T	F	T	T
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	F	T	T	T
F	F	F	F	F	F	F

Table 12.16

The columns corresponding to $(p \vee q) \vee r$ and $p \vee (q \vee r)$ are identical.

Hence $p \vee (q \vee r) \equiv (p \vee q) \vee r$.

Similarly, (ii) $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$ can be proved.

4. Distributive Laws

(i) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ (ii) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

Proof (i)

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

Table 12.17

The columns corresponding to $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are identical. Hence $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$.

Similarly (ii) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ can be proved.

5. Identity Laws

(i) $p \vee \mathbb{T} \equiv \mathbb{T}$ and $p \vee \mathbb{F} \equiv p$ (ii) $p \wedge \mathbb{T} \equiv p$ and $p \wedge \mathbb{F} \equiv \mathbb{F}$

p	$\mathbb{T}$	$\mathbb{F}$	$p \vee \mathbb{T}$	$p \vee \mathbb{F}$
T	T	F	T	T
F	T	F	T	F

Table12.18

(i) The entries in the columns corresponding to $p \vee \mathbb{T}$ and $\mathbb{T}$ are identical and hence they are equivalent. The entries in the columns corresponding to $p \vee \mathbb{F}$ and p are identical and hence they are equivalent.

Dually

(ii) $p \wedge \mathbb{T} \equiv p$ and $p \wedge \mathbb{F} \equiv \mathbb{F}$ can be proved.

6. Complement Laws

(i) $p \vee \neg p \equiv \mathbb{T}$ and $p \wedge \neg p \equiv \mathbb{F}$ (ii) $\neg \mathbb{T} \equiv \mathbb{F}$ and $\neg \mathbb{F} \equiv \mathbb{T}$

Proof

p	$\neg p$	$\mathbb{T}$	$\neg \mathbb{T}$	$\mathbb{F}$	$\neg \mathbb{F}$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F	F	T	T	F
F	T	T	F	F	T	T	F

Table 12.19

(i) The entries in the columns corresponding to $p \vee \neg p$ and $\mathbb{T}$ are identical and hence they are equivalent. The entries in the columns corresponding to $p \wedge \neg p$ and $\mathbb{F}$ are identical and hence they are equivalent.

(ii) The entries in the columns corresponding to $\neg \mathbb{T}$ and $\mathbb{F}$ are identical and hence they are equivalent. The entries in the columns corresponding to $\neg \mathbb{F}$ and $\mathbb{T}$ are identical and hence they are equivalent.

7. Involution Law or Double Negation Law

$\neg(\neg p) \equiv p$

Proof

p	$\neg p$	$\neg(\neg p)$
T	F	T
F	T	F

Table 12.20

The entries in the columns corresponding to $\neg(\neg p)$ and p are identical and hence they are equivalent.

8. de Morgan's Laws

(i) $\neg(p \wedge q) \equiv \neg p \vee \neg q$ (ii) $\neg(p \vee q) \equiv \neg p \wedge \neg q$

Proof of (i)

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg(p \wedge q)$	$\neg p \vee \neg q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

Table 12.21

The entries in the columns corresponding to $\neg(p \wedge q)$ and $\neg p \vee \neg q$ are identical and hence they are equivalent. Therefore $\neg(p \wedge q) \equiv \neg p \vee \neg q$. Dually (ii) $\neg(p \vee q) \equiv \neg p \wedge \neg q$ can be proved.

9. Absorption Laws

(i) $p \vee (p \wedge q) \equiv p$ (ii) $p \wedge (p \vee q) \equiv p$

p	q	$p \wedge q$	$p \vee q$	$p \vee (p \wedge q)$	$p \wedge (p \vee q)$
T	T	T	T	T	T
T	F	F	T	T	T
F	T	F	T	F	F
F	F	F	F	F	F

Table 12.22

(i) The entries in the columns corresponding to $p \vee (p \wedge q)$ and p are identical and hence they are equivalent.

(ii) The entries in the columns corresponding to $p \wedge (p \vee q)$ and p are identical and hence they are equivalent.

Example 12.17

Establish the equivalence property: $p \to q \equiv \neg p \vee q$

Solution

p	q	$\neg p$	$p \to q$	$\neg p \vee q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Table 12.23

The entries in the columns corresponding to $p \to q$ and $\neg p \vee q$ are identical and hence they are equivalent.

Example 12.18

Establish the equivalence property connecting the bi-conditional with conditional:

$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

Solution

p	q	$p \to q$	$q \to p$	$p \leftrightarrow q$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

Table 12.24

The entries in the columns corresponding to $p \leftrightarrow q$ and $(p \to q) \wedge (q \to p)$ are identical and hence they are equivalent.

Example 12.19

Using the equivalence property, show that $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$.

Solution

It can be obtained by using examples 12.15 and 12.16 that

$$p \leftrightarrow q \equiv (\neg p \vee q) \wedge (\neg q \vee p) \quad \text{... (1)}$$

$$\equiv (\neg p \vee q) \wedge (p \vee \neg q) \text{ (by Commutative Law)} \quad \text{... (2)}$$

$$\equiv (\neg p \wedge (p \vee \neg q)) \vee (q \wedge (p \vee \neg q)) \text{ (by Distributive Law)}$$

$$\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) \vee (q \wedge p) \vee (q \wedge \neg q) \text{ (by Distributive Law)}$$

$$\equiv \mathbb{F} \vee (\neg p \wedge \neg q) \vee (q \wedge p) \vee \mathbb{F}; \text{ (by Complement Law)}$$

$$\equiv (\neg p \wedge \neg q) \vee (q \wedge p); \text{ (by Identity Law)}$$

$$\equiv (p \wedge q) \vee (\neg p \wedge \neg q); \text{ (by Commutative Law)}$$

Finally (1) becomes $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$.

EXERCISE 12.2

1. Let p : Jupiter is a planet and q : India is an island be any two simple statements. Give verbal sentence describing each of the following statements.
   (i) $\neg p$ (ii) $p \wedge \neg q$ (iii) $\neg p \vee q$ (iv) $p \rightarrow \neg q$ (v) $p \leftrightarrow q$
2. Write each of the following sentences in symbolic form using statement variables p and q.
   (i) 19 is not a prime number and all the angles of a triangle are equal.
   (ii) 19 is a prime number or all the angles of a triangle are not equal
   (iii) 19 is a prime number and all the angles of a triangle are equal
   (iv) 19 is not a prime number
3. Determine the truth value of each of the following statements
   (i) If $6+2=5$, then the milk is white.
   (ii) China is in Europe or $\sqrt{3}$ is an integer
   (iii) It is not true that $5+5=9$ or Earth is a planet
   (iv) 11 is a prime number and all the sides of a rectangle are equal
4. Which one of the following sentences is a proposition?
   (i) $4+7=12$ (ii) What are you doing? (iii) $3^n \le 81, n \in \mathbb{N}$
   (iv) Peacock is our national bird (v) How tall this mountain is!
5. Write the converse, inverse, and contrapositive of each of the following implication.
   (i) If x and y are numbers such that $x = y$, then $x^2 = y^2$
   (ii) If a quadrilateral is a square then it is a rectangle
6. Construct the truth table for the following statements.
   (i) $\neg p \wedge \neg q$ (ii) $\neg(p \wedge \neg q)$ (iii) $(p \vee q) \vee \neg q$ (iv) $(\neg p \rightarrow r) \wedge (p \leftrightarrow q)$

7. Verify whether the following compound propositions are tautologies or contradictions or contingency

   (i) $(p \wedge q) \wedge \neg(p \vee q)$ (ii) $((p \vee q) \wedge \neg p) \rightarrow q$

   (iii) $(p \rightarrow q) \leftrightarrow (\neg p \rightarrow q)$ (iv) $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$

8. Show that (i) $\neg(p \wedge q) \equiv \neg p \vee \neg q$ (ii) $\neg(p \rightarrow q) \equiv p \wedge \neg q$.

9. Prove that $q \rightarrow p \equiv \neg p \rightarrow \neg q$

10. Show that $p \rightarrow q$ and $q \rightarrow p$ are not equivalent

11. Show that $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

12. Check whether the statement $p \rightarrow (q \rightarrow p)$ is a tautology or a contradiction without using the truth table.

13. Using truth table check whether the statements $\neg(p \vee q) \vee (\neg p \wedge q)$ and $\neg p$ are logically equivalent.

14. Prove $p \rightarrow (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$ without using truth table.

15. Prove that $p \rightarrow (\neg q \vee r) \equiv \neg p \vee (\neg q \vee r)$ using truth table.

EXERCISE 12.3

Choose the correct or the most suitable answer from the given four alternatives.

1. A binary operation on a set S is a function from

   (1) $S \rightarrow S$ (2) $(S \times S) \rightarrow S$ (3) $S \rightarrow (S \times S)$ (4) $(S \times S) \rightarrow (S \times S)$

2. Subtraction is not a binary operation in

   (1) $\mathbb{R}$ (2) $\mathbb{Z}$ (3) $\mathbb{N}$ (4) $\mathbb{Q}$

3. Which one of the following is a binary operation on $\mathbb{N}$?

   (1) Subtraction (2) Multiplication (3) Division (4) All the above

4. In the set $\mathbb{R}$ of real numbers '$*$' is defined as follows. Which one of the following is not a binary operation on $\mathbb{R}$?

   (1) $a * b = \min(a \cdot b)$ (2) $a * b = \max(a, b)$

   (3) $a * b = a$ (4) $a * b = a^b$

5. The operation $*$ defined by $a * b = \dfrac{ab}{7}$ is not a binary operation on

   (1) $\mathbb{Q}^+$ (2) $\mathbb{Z}$ (3) $\mathbb{R}$ (4) $\mathbb{C}$

6. In the set $\mathbb{Q}$ define $a \odot b = a + b + ab$. For what value of y, $3 \odot (y \odot 5) = 7$?

   (1) $y = \dfrac{2}{3}$ (2) $y = \dfrac{-2}{3}$ (3) $y = \dfrac{-3}{2}$ (4) $y = 4$

7. If $a * b = \sqrt{a^2 + b^2}$ on the real numbers then $*$ is

   (1) commutative but not associative (2) associative but not commutative

   (3) both commutative and associative (4) neither commutative nor associative

8. Which one of the following statements has the truth value T ?

(1) $\sin x$ is an even function.

(2) Every square matrix is non-singular

(3) The product of complex number and its conjugate is purely imaginary

(4) $\sqrt{5}$ is an irrational number

9. Which one of the following statements has truth value F ?

(1) Chennai is in India or $\sqrt{2}$ is an integer

(2) Chennai is in India or $\sqrt{2}$ is an irrational number

(3) Chennai is in China or $\sqrt{2}$ is an integer

(4) Chennai is in China or $\sqrt{2}$ is an irrational number

10. If a compound statement involves 3 simple statements, then the number of rows in the truth table is

(1) 9 (2) 8 (3) 6 (4) 3

11. Which one is the inverse of the statement $(p \vee q) \rightarrow (p \wedge q)$?

(1) $(p \wedge q) \rightarrow (p \vee q)$ (2) $\neg(p \vee q) \rightarrow (p \wedge q)$

(3) $(\neg p \vee \neg q) \rightarrow (\neg p \wedge \neg q)$ (4) $(\neg p \wedge \neg q) \rightarrow (\neg p \vee \neg q)$

12. Which one is the contrapositive of the statement $(p \vee q) \rightarrow r$?

(1) $\neg r \rightarrow (\neg p \wedge \neg q)$ (2) $\neg r \rightarrow (p \vee q)$

(3) $r \rightarrow (p \wedge q)$ (4) $p \rightarrow (q \vee r)$

13. The truth table for $(p \wedge q) \vee \neg q$ is given below

p	q	$(p \wedge q) \vee (\neg q)$
T	T	(a)
T	F	(b)
F	T	(c)
F	F	(d)

Which one of the following is true?

	(a)	(b)	(c)	(d)
(1)	T	T	T	T
(2)	T	F	T	T
(3)	T	T	F	T
(4)	T	F	F	F

14. In the last column of the truth table for $\neg(p \vee \neg q)$ the number of final outcomes of the truth value 'F' are

(1) 1 (2) 2 (3) 3 (4) 4

15. Which one of the following is incorrect? For any two propositions p and q, we have

(1) $\neg(p \vee q) \equiv \neg p \wedge \neg q$ (2) $\neg(p \wedge q) \equiv \neg p \vee \neg q$

(3) $\neg(p \vee q) \equiv \neg p \vee \neg q$ (4) $\neg(\neg p) \equiv p$

16.

p	q	$(p \wedge q) \rightarrow \neg p$
T	T	(a)
T	F	(b)
F	T	(c)
F	F	(d)

Which one of the following is correct for the truth value of $(p \wedge q) \rightarrow \neg p$?

	(a)	(b)	(c)	(d)
(1)	T	T	T	T
(2)	F	T	T	T
(3)	F	F	T	T
(4)	T	T	T	F

17. The dual of $\neg(p \vee q) \vee [p \vee (p \wedge \neg r)]$ is

(1) $\neg(p \wedge q) \wedge [p \vee (p \wedge \neg r)]$ (2) $(p \wedge q) \wedge [p \wedge (p \vee \neg r)]$

(3) $\neg(p \wedge q) \wedge [p \wedge (p \wedge r)]$ (4) $\neg(p \wedge q) \wedge [p \wedge (p \vee \neg r)]$

18. The proposition $p \wedge (\neg p \vee q)$ is

(1) a tautology (2) a contradiction

(3) logically equivalent to $p \wedge q$ (4) logically equivalent to $p \vee q$

19. Determine the truth value of each of the following statements:

(a) $4+2=5$ and $6+3=9$ (b) $3+2=5$ and $6+1=7$

(c) $4+5=9$ and $1+2=4$ (d) $3+2=5$ and $4+7=11$

	(a)	(b)	(c)	(d)
(1)	F	T	F	T
(2)	T	F	T	F
(3)	T	T	F	F
(4)	F	F	T	T

20. Which one of the following is not true?

(1) Negation of a negation of a statement is the statement itself.

(2) If the last column of the truth table contains only T then it is a tautology.

(3) If the last column of its truth table contains only F then it is a contradiction

(4) If p and q are any two statements then $p \leftrightarrow q$ is a tautology.

SUMMARY

(1) A **binary operation** $*$ on a non-empty set S is a rule, which associates to each ordered pair (a,b) of elements a,b in S an unique element $a*b$ in S.

(2) **Commutative property:** Any binary operation $*$ defined on a nonempty set S is said to satisfy the commutative property, if $a*b=b*a,\ \forall a,b\in S$.

(3) **Associative property:** Any binary operation $*$ defined on a nonempty set S is said to satisfy the associative property, if $a*(b*c)=(a*b)*c,\ \forall a,b,c\in S$.

(4) **Existence of identity property:** An element $e\in S$ is said to be the **Identity Element** of S under the binary operation $*$ if for all $a\in S$ we have that $a*e=a$ and $e*a=a$.

(5) **Existence of inverse property:** If an identity element e exists and if for every $a\in S$, there exists b in S such that $a*b=e$ and $b*a=e$ then $b\in S$ said to be the **Inverse Element** of a. In such instance, we write $b=a^{-1}$.

(6) **Uniqueness of Identity:** In an algebraic structure the identity element (if exists) must be unique.

(7) **Uniqueness of Inverse:** In an algebraic structure the inverse of an element (if exists) must be unique.

(8) A **Boolean Matrix** is a real matrix whose entries are either 0 or 1.

(9) **Modular arithmetic:** Let n be a positive integer greater than 1 called the **modulus**. We say that two integers a and b are congruent modulo n if $b-a$ is divisible by n. In other words $a\equiv b$ (mod n) means $a-b=n\cdot k$ for some integer k and b is the **least non-negative integer reminder** when a is divided by n. $(0\le b\le n-1)$

(10) Mathematical logic is a study of reasoning through mathematical symbols.

(11) Let p be a simple statement. Then the **negation** of p is a statement whose truth value is opposite to that of p. It is denoted by $\neg p$, read as **not** p. The truth value of $\neg p$ is T, if p is F, otherwise it is F.

(12) Let p and q be any two simple statements. The **conjunction** of p and q is obtained by connecting p and q by the word **and**. It is denoted by $p\wedge q$, read as 'p conjunction q' or 'p hat q'. The truth value of $p\wedge q$ is T, whenever both p and q are T and it is F otherwise.

(13) The **disjunction** of any two simple statements p and q is the compound statement obtained by connecting p and q by the word 'or'. It is denoted by $p\vee q$, read as 'p disjunction q' or 'p cup q'. The truth value of $p\vee q$ is F, whenever both p and q are F and it is T otherwise.

(14) The **conditional statement** of any two statements p and q is the statement, 'If p, then q' and it is denoted by $p\rightarrow q$. The statement $p\rightarrow q$ has a truth value F when q has the truth value F and p has the truth value T; otherwise it has the truth value T.

(15) The **bi-conditional statement** of any two statements p and q is the statement 'p if and only if q' and is denoted by $p\leftrightarrow q$ The statement $p\leftrightarrow q$ has the truth value T whenever both p and q have identical truth values; otherwise has the truth value F.

(16) A statement is said to be a **tautology** if its truth value is always T irrespective of the truth values of its component statements. It is denoted by $\mathbb{T}$.

(17) A statement is said to be a **contradiction** if its truth value is always F irrespective of the truth values of its component statements. It is denoted by $\mathbb{F}$.

(18) A statement which is neither a tautology nor a contradiction is called **contingency**.

(19) Any two compound statements A and B are said to be **logically equivalent** or simply **equivalent** if the columns corresponding to A and B in the truth table have **identical truth values**. The logical equivalence of the statements A and B is denoted by $A \equiv B$ or $A \Leftrightarrow B$. Further note that if A and B are logically equivalent, then $A \leftrightarrow B$ must be a **tautology**.

(20) **Some laws of equivalence:**

Idempotent Laws: (i) $p \vee p \equiv p$ (ii) $p \wedge p \equiv p$.

Commutative Laws: (i) $p \vee q \equiv q \vee p$ (ii) $p \wedge q \equiv q \wedge p$.

Associative Laws: (i) $p \vee (q \vee r) \equiv (p \vee q) \vee r$ (ii) $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$.

Distributive Laws: (i) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

(ii) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

Identity Laws: (i) $p \vee \mathbb{T} \equiv \mathbb{T}$ and $p \vee \mathbb{F} \equiv p$

(ii) $p \wedge \mathbb{T} \equiv p$ and $p \wedge \mathbb{F} \equiv \mathbb{F}$

Complement Laws : (i) $p \vee \neg p \equiv \mathbb{T}$ and $p \wedge \neg p \equiv \mathbb{F}$

(ii) $\neg \mathbb{T} \equiv \mathbb{F}$ and $\neg \mathbb{F} \equiv \mathbb{T}$

Involution Law or Double Negation Law: $\neg(\neg p)\neg p$

de Morgan's Laws: (i) $\neg(p \wedge q) \equiv \neg p \vee \neg q$ (ii) $\neg(p \vee q) \equiv \neg p \vee \neg q$

Absorption Laws: (i) $p \vee (p \wedge q) \equiv p$ (ii) $p \wedge (p \vee q) \equiv p$

ICT CORNER

https://ggbm.at/dy9kwgbt or Scan the QR Code

Open the Browser, type the URL Link given (or) Scan the QR code. GeoGebra work book named "**12th Standard Mathematics Vol-2**" will open. In the left side of work book there are chapters related to your text book. Click on the chapter named "**Discrete Mathematics**". You can see several work sheets related to the chapter. Go through all the work sheets.

B262_12_MATHS_EM

ANSWERS

Chapter 7

EXERCISE 7.1

(1) (i) 21 m/s (ii) 15 m/s and 27 m/s

(2) (2)(i) 5sec (ii) 128 ft/s (iii) 160 ft/s

(3) (i) 1,2 sec (ii) 34 m (iii) $-6\,m/s^2$, $6\,m/s^2$

(4) 75 units (5) $\frac{1}{2}$ kg/m, $\frac{1}{6}$ kg/m

(6) 20π sq.cm/s (7) 2π km/s (8) $\frac{9}{10\pi}$ m/min

(9) (i) $\frac{-8}{3}$ m/s (ii) 26.83 sq.m/sec (10) 70 km/hr.

EXERCISE 7.2

(1) (i) 7 (ii) ∞ (2) $(1,0)$

(3) $(0,3)$ and $(4,-25)$ (4) $(2,-1)$ and $(-2,1)$

(5) (i) $2x+y=2\,;\,x-2y=1$ (ii) $2x-y=-2\,;\,x+2y=4$

(iii) $x-y=0\,;\,x+y=\pi$ (iv) $4x+2y=5\,;\,2x-4y=-5$

(6) $12x-y=15\,;\,12x-y=-17$ (7) $x+2y=7\,;\,x+2y=-1$

(8) $(2\cos t)x+(7\sin t)y=14\,;\,(7\sin t)x-(2\cos t)y=45\sin t\cos t$ (9) $\tan^{-1}(3)$

EXERCISE 7.3

(1) (i) not continuous at $x=0$ (ii) not continuous at $x=\frac{\pi}{2}$ (iii) $f(2)\neq f(7)$

(2) (i) $\frac{1}{2}$ (ii) $-2+2\sqrt{2}$ (iii) $\frac{9}{4}$

(3) (i) not continuous at $x=0$ (ii) not differentiable at $x=\frac{-1}{3}$

(4) (i) $\pm\frac{2}{\sqrt{3}}$ (ii) 7

(6) 320 km (8) No. Since $f'(x)$ cannot be 2.5 at any point in $(0,2)$.

EXERCISE 7.4

(1) (i) $e^x=1+\frac{x}{\lfloor 1}+\frac{x^2}{\lfloor 2}+\cdots$ (ii) $\sin x=x-\frac{x^3}{\lfloor 3}+\frac{x^5}{\lfloor 5}-\frac{x^7}{\lfloor 7}+\cdots$

(iii) $\cos x=1-\frac{x^2}{\lfloor 2}+\frac{x^4}{\lfloor 4}-\frac{x^6}{\lfloor 6}+\cdots$

(iv) $\log(1-x)=-\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots\right)$

(v) $\tan^{-1}(x)=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots$ (vi) $\cos^2 x=1-\frac{2x^2}{\lfloor 2}+\frac{2^3x^4}{\lfloor 4}-\frac{2^5x^6}{\lfloor 6}+\cdots$

(2) $\log x=(x-1)-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3-\frac{1}{4}(x-1)^4+\cdots$

(3) $\frac{\sqrt{2}}{2}\left(1+\frac{1}{\lfloor 1}\left(x-\frac{\pi}{4}\right)-\frac{1}{\lfloor 2}\left(x-\frac{\pi}{4}\right)^2-\frac{1}{\lfloor 3}\left(x-\frac{\pi}{4}\right)^3+\cdots\right)$

(4) $f(x)=-(x-1)+(x-1)^2$

EXERCISE 7.5

(1) $\frac{1}{2}$ (2) 2 (3) ∞ (4) 1 (5) 0 (6) 0

(7) $\frac{-3}{2}$ (8) 1 (9) e (10) 1 (11) $\frac{1}{\sqrt{e}}$

EXERCISE 7.6

(1) (i) absolute maximum = -1, absolute minimum = -26

(ii) absolute maximum = 16, absolute minimum = -1

(iii) absolute maximum = 9, absolute minimum = $-\frac{9}{8}$

(iv) absolute maximum = $\frac{3\sqrt{3}}{2}$, absolute minimum = 0

(2) (i) strictly increasing on $(-\infty,-2)$ and $(1,\infty)$, strictly decreasing on $(-2,1)$

local maximum $=20$ local minimum $=-7$

(ii) strictly decreasing on $(-\infty,5)$ and $(5,\infty)$. No local extremum.

(iii) strictly increasing on $(-\infty,\infty)$. No local extremum.

(iv) strictly decreasing on $(0,1)$, strictly increasing on $(1,\infty)$. local minimum $=\frac{1}{3}$

(v) strictly increasing on $\left(0,\frac{\pi}{4}\right),\left(\frac{3\pi}{4},\frac{5\pi}{4}\right)$, and $\left(\frac{7\pi}{4},2\pi\right)$.

strictly decreasing on $\left(\frac{\pi}{4},\frac{3\pi}{4}\right)$ and $\left(\frac{5\pi}{4},\frac{7\pi}{4}\right)$. local maximum$=\frac{11}{2}$ at $x=\frac{\pi}{4}$, $\frac{5\pi}{4}$.

local minimum$=\frac{9}{2}$ at $x=\frac{3\pi}{4}$, $\frac{7\pi}{4}$.

EXERCISE 7.7

(1) (i) concave upwards on $(-\infty,2)$ and $(4,\infty)$. Concave downwards on $(2,4)$

Points of inflection $(2,-16)$ and $(4,0)$

(ii) concave upwards on $\left(\frac{3\pi}{4},\frac{7\pi}{4}\right)$. Concave downwards on $\left(0,\frac{3\pi}{4}\right)$ and $\left(\frac{7\pi}{4},2\pi\right)$

Points of inflection $\left(\frac{3\pi}{4},0\right)$ and $\left(\frac{7\pi}{4},0\right)$

(iii) concave upwards on $(0,\infty)$. Concave downward on $(-\infty,0)$

Points of inflection $(0,0)$

(2) (i)local minimum = -2 ; local maximum $=2$ (ii)local minimum = $-\frac{1}{e}$

(iii)local minimum = 0 ; local maximum $=\frac{1}{e^2}$

(3) strictly increasing on $(-\infty,-1)$ and $\left(\frac{1}{2},\infty\right)$. strictly increasing on $\left(-1,\frac{1}{2}\right)$

local maximum = 6, local minimum = $-\frac{3}{4}$

concave downwards on $\left(-\infty,-\frac{1}{4}\right)$; concave upwards on $\left(-\frac{1}{4},\infty\right)$.

point of inflection $\left(-\frac{1}{4},\frac{21}{8}\right)$

EXERCISE 7.8

(1) $6, 6$ (2) $2\sqrt{5}, 2\sqrt{5}$ (3) 50 (4) 100m^2 (5) $9\text{cm}, 6\text{cm}$ (6) 1200m

(7) $10\sqrt{2}, 10\sqrt{2}$ (9) $\sqrt{2}r, \frac{r}{\sqrt{2}}$ (10) 6cm, 6cm, 3cm (11) $32\pi, 0$

EXERCISE 7.9

(1) (i) $x=-1, x=1, y=1$ (ii) $x=-1, y=x-1$, (iii) $y=-3, y=3$

(iv) $y=x-9, x=-3$

EXERCISE 7.10

1	2	3	4	5	6	7	8	9	10
(1)	(2)	(2)	(2)	(1)	(2)	(3)	(4)	(3)	(1)
11	12	13	14	15	16	17	18	19	20
(3)	(4)	(3)	(4)	(2)	(3)	(3)	(1)	(4)	(3)

Chapter 8

Exercise 8.1

1. (i) 3.0074 2. (i) 24.73 (ii) 1.9688 (iii) 2.963

3. (i) $7x-4$ (ii) $\frac{9-4x}{5}$ (iii) $\frac{x+1}{4}$

4. (i)0.0225π cm^2, (ii) $0.006\,\text{cm}^2$ (iii) 0.6%

5. (i) Volume decreases by $80\pi \; cm^3$ (ii) Surface area decreases by $16\pi \; cm^2$ 6. 1%

Exercise 8.2

1. (i) $\frac{2(1-2x)^2(8x-7)}{(3-4x)^2}dx$ (ii) $\frac{4}{3}\frac{\cos 2x}{(3+\sin 2x)^{\frac{1}{3}}}dx$ (iii) $e^{x^2-5x+7}[(2x-5)\cos(x^2-1)-2x\sin(x^2-1)]dx$

2. (i) 0.7 (ii) 0.18 3. (i) $\Delta f = 3.125$, $df = 2.0$ (ii) $\Delta f = 0.11$, $df = 0.1$

4. 3.0013029 5. (i) $\frac{6}{\pi}$ cm (ii) $\frac{40}{\pi}\%$ 6. $30\pi \; mm^3$ 7. $0.4\pi \; mm^2$ 8. 8000

9. (i) $\simeq 3$ words (ii) $\simeq 1$ word 10. 5.25π, 4.76% 11. $60 \; cm^3$, $61.2 \; cm^3$

Exercise 8.3

1. $\frac{1}{8}$ 2. 1 4. $\cos(1)$

Exercise 8.4

1. (i) 27, -14 (ii) 11, -4 (iii) 2, 0, 4 (iv) $e^2((\log 2)-1)$, $e^2(1+\log 8)$

3. $\frac{x^2-y^2}{x^2y}, \frac{y^2-x^2}{y^2x} + 3z^2$, 6yz 4. $\frac{3(x^2+y^2+z^2)}{(x^3+y^3+z^3)}$

5. (i) e^y+6x, $6y$, xe^y, e^y+6x

(ii) $\frac{-15}{(5x+3y)^2}, \frac{-25}{(5x+3y)^2}, \frac{-9}{(5x+3y)^2}, \frac{-15}{(5x+3y)^2}$ (iii) 3, $2-25\cos 5x$, 0, 3

10. (i) $72x+84y+0.04xy-0.05x^2-0.05y^2-2000$ (ii) 24, -48, Keeping y constant and increasing x increases profit.

Exercise 8.5

1. $6x-7y-7$ 2. $-(x+20y+16)$ 3. $(2x-y)dx+\left(-x+\frac{1}{2}y\right)dy$

4. $(y+z)dx+(x+z)dy+(y+x)dz$

Exercise 8.6

1. $e^t\left(2e^t\sin t+3\sin^4 t+e^t\cos t+12\sin^3 t\cos t\right)$, 1

2. $\left(1+e^{2t}\right)^2\left[\cos^3 t\left(1+e^{2t}\right)-\sin t\sin 2t\left(1+e^{2t}\right)+6e^{2t}\sin t\cos^2 t\right]$

3. $4e^{2t}$ 4. $-e^{-2t}\left[\sin 2t-\cos 2t\right]$ 5. $18e^{3s}-3e^s\cos s+3e^s\sin s-4\sin s\cos s$, 15

6. $\frac{3e}{1+e^2}+2\tan^{-1}e$, $\frac{e}{1+e^2}$

7. $te^{st^2}\left[t\sin\left(s^2t\right)+2s\cos\left(s^2t\right)\right]$, $\frac{du}{dt}=se^{st^2}\left[2t+\sin\left(s^2t\right)+s\cos\left(s^2t\right)\right]$, $e\left[\sin(1)+2\cos(1)\right]$, $e\left[2\sin(1)+\cos(1)\right]$

8. $3s^3(e^{3t}+s^2e^{-t})$, $3s^2e^t(e^{2t}-5e^{-2t}s^2)$ 9. $2u(1+2v)$, $2\left(u^2-v\right)$, 3, $\frac{-3}{2}$

Exercise 8.7

1. (i) not homogeneous (ii) Homogeneous, deg.3
   (iii) homogeneous, deg.0 (iv) not homogeneous 6. 5

Exercise 8.8

1	2	3	4	5	6	7	8
(2)	(2)	(2)	(4)	(3)	(2)	(4)	(2)
9	10	11	12	13	14	15	
(3)	(1)	(2)	(3)	(2)	(4)	(1)	

Chapter 9

Exercise 9.1

1. 0.6 2. 0.855 3. 0.375

Exercise 9.2

1. (i) $\frac{13}{2}$ (ii) $\frac{25}{3}$

Exercise 9.3

1. (i) $\frac{1}{4}\log\frac{5}{3}$ (ii) $\frac{\pi}{8}$ (iii) $\frac{\pi}{2}-1$ (iv) $e^{\frac{\pi}{2}}$ (v) $\frac{8}{21}$ (vi) $\frac{1}{2}$

2. (i) 0 (ii) π (iii) $\frac{\pi-2}{4}$ (iv) 0 (v) 0 (vi) $\frac{13}{10}$ (vii) $\frac{\pi}{4}$
   (viii) $\frac{\pi}{8}\log 2$ (ix) $\frac{\pi}{2}(\pi-2)$ (x) $\frac{\pi}{8}$ (xi) $\frac{\pi^2}{2}$

Exercise 9.4

1. $\frac{3}{8}-\frac{19}{8}e^{-2}$ 2. $\frac{1}{\sqrt{2}}\left(\frac{\pi}{12}+\frac{1}{9}\right)$ 3. $1+e^{\frac{\pi}{4}}\left[\frac{\pi}{4}-1\right]$ 4. $-\frac{\pi}{4}$

Exercise 9.5

1. (i) $\frac{\pi}{2\sqrt{6}}$ (ii). $\frac{\pi}{6\sqrt{5}}$

Exercise 9.6

1. (i) $\frac{63\pi}{512}$ (ii) $\frac{16}{35}$ (iii) $\frac{5\pi}{64}$ (iv) $\frac{8}{45}$ (v) $\frac{\pi}{32}$ (vi) $\frac{64}{35}$ (vii) $\frac{1}{24}$ (viii) $\frac{1}{60}$

Exercise 9.7

1. (i) $\frac{5!}{3^6}$ (ii) 29 (2) $\frac{1}{8}$

Exercise 9.8

1. 7.5 2. 2 3. 15 4. 36 5. $2\sqrt{2}$ 6. $\log 2$ 7. $\frac{9}{2}$ 8. yes, $\frac{16}{3}$ 9. $\frac{4}{3}$ 10. $\frac{4}{3}\left(4\pi+\sqrt{3}\right)$

Exercise 9.9

1. $\frac{4\pi}{5}$ 2. $\frac{\pi}{4}\left[1-e^{-4}\right]$ 3. 8π 4. $\frac{2\pi}{15}$ 5. $\frac{14}{3}\pi m^3$ 6. $\frac{1000}{3}\pi cm^3$

Exercise 9.10

1	2	3	4	5	6	7	8	9	10
(1)	(3)	(3)	(4)	(4)	(3)	(3)	(3)	(2)	(1)
11	12	13	14	15	16	17	18	19	20
(4)	(2)	(2)	(4)	(4)	(4)	(3)	(4)	(2)	(1)

Chapter 10

Exercise 10.1

1. (i) 1,1 (ii) 3,2 (iii) 2, does not exist (iv) 1, 2 (v) 1,4

(vi) 2,2 (vii) 2,6 (viii) 2, does not exist (ix) 3,1 (x) 1, 1

Exercise 10.2

1. (i) $\frac{dQ}{dt}=kQ$ (ii) $\frac{dP}{dt}=kP(500000-P)$ (iii) $\frac{dP}{dT}=\frac{kP}{T^2}$ (iv) $\frac{dx}{dt}=\frac{2x}{25}+400$ 2. $\frac{dr}{dt}=-k$

Exercise 10.3

1. (i) $\frac{d^2y}{dx^2}=0$ (ii) $\frac{d^2x}{dy^2}=0$ 2. $r^2\left[1+\left(\frac{dy}{dx}\right)^2\right]=\left(x\frac{dy}{dx}-y\right)^2$

3. $x^2+2xy\frac{dy}{dx}-y^2=0$ 4. $2ay''+y'^3=0$ 5. $xy'-2y-2=0$ 6. $xy'^2+xyy''-yy'=0$

7. $\frac{d^2y}{dx^2}=64y$ 8. $xy''+2y'+x^2-xy-2=0$

Exercise 10.4

2. (i) $m=-2$ (ii) $m=2,3$ 3. $2y^2=x+48$

Exercise 10.5

1. $F=(F-kV)e^{\frac{kt}{M}}$ 2. $k^2\left(1-e^{-\frac{2gx}{k^2}}\right)=v^2$ 3. $y=\frac{1-x}{1+x}$

4. (i) $\sin^{-1}y=\sin^{-1}x+C$ (ii) $y\tan^{-1}x=C$ (iii) $\sin\left(\frac{y-1}{x}\right)=a$ (iv) $e^x+e^{-y}+\frac{x^4}{4}=C$

(v) $\left(e^y+1\right)\sin x=C$ (vi) $\sin\left(\frac{x}{y}\right)=e^{nx+c}$ (vii) $3y=-\left(25-x^2\right)^{\frac{3}{2}}+3C$ (viii) $\sin y=e^x\log x+C$

(ix) $\sec y = 2\sin x + C$ (x) $\frac{1}{2}\left[(x+y)+\sin(x+y)\cos(x+y)\right] = x + C$

Exercise 10.6

1. $\sin\left(\frac{y}{x}\right) = \log|Cx|$ 2. $y = Ce^{\frac{x^3}{3y^3}}$ 3. $e^{\frac{x}{y}} = \log|Cy|$ 4. $3x^2y + 2y^3 = C$

5. $xy^2 - x^2y = C$ 6. $C = xe^{\tan\left(\frac{y}{x}\right)}$ 7. $y + 3xe^{\frac{y}{x}} = 3$ 8. $x_0 = \pm\sqrt{3}e$

Exercise 10.7

1. $y = \sin x + C\cos x$ 2. $y = \frac{\sin^{-1}x}{\sqrt{1-x^2}} + C(1-x^2)^{-\frac{1}{2}}$ 3. $(y+\cos x)x = \sin x + C$

4. $y(x^2+1) = \frac{x}{2}\sqrt{x^2+4} + \frac{1}{2}\log\left|x+\sqrt{x^2+4}\right| + C$ 5. $xy^2 = 2y^5 + C$

6. $xy\sin x + \cos x = C$ 7. $ye^{\sin^{-1}x} = \frac{e^{2\sin^{-1}x}}{2} + C$ 8. $y\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right) = x + \frac{2}{3}x\sqrt{x} + C$

9. $xy + \tan^{-1}y = C$ 10. $y\log x + \frac{\cos 2x}{2} = C$ 11. $2y = (x+a)^4 + 2C(x+a)^2$

12. $y(1+x^3) = \frac{x}{2} - \frac{\sin 2x}{4} + C$ 13. $4yx = 2x^2\log x - x^2 + 4C$ 14. $x^2y = \frac{x^4}{4}\log x - \frac{x^4}{16} + C$

15. $2x^3y = x^2 + 3$

Exercise 10.8

1. After 10 hours the number of bacteria as 9 times the original number of bacteria.

2. $P = 300000\left(\frac{4}{3}\right)^{\frac{t}{40}}$ 3. $i = Ce^{-\frac{Rt}{L}}$ 4. $v = \frac{10}{e^2}$ 5. $P = 10000e^{0.075}$

6. $\frac{9^{10}}{10^8}$ % of the radioactive element will remain after 1000 years.

7. (i) $65.33°C$ (ii) 51.91 *mts*

8. (i) $T \simeq 151°F$ (ii) $t = 22.523$. She drunk the coffee between 10.22 and 10.30 approximately.

9. $20°$ 10. $x = 100\left(1 - e^{\frac{-3t}{50}}\right)$

Exercise 10.9

Q	1	2	3	4	5	6	7	8	9	10
A	(1)	(2)	(3)	(2)	(2)	(3)	(3)	(2)	(2)	(3)
Q	11	12	13	14	15	16	17	18	19	20
A	(3)	(3)	(1)	(1)	(2)	(3)	(2)	(4)	(2)	(4)
Q	21	22	23	24	25					
A	(1)	(1)	(2)	(2)	(1)					

Chapter 11

EXERCISE 11.1

(1)

Values of Random Variable	0	1	2	3	Total
Number of points in inverse image	1	3	3	1	8

(2)

Values of Random Variable	0	1	2	Total
Number of points in inverse image	325	676	325	1326

(3)

Values of Random Variable	0	1	2	3	Total
Number of points in inverse image	4	30	40	10	84

(4)

Values of Random Variable	–20	5	30	Total
Number of points in inverse image	28	48	15	91

(5)

Values of Random Variable	4	5	6	7	8	Total
Number of points in inverse image	1	4	10	12	9	36

EXERCISE 11.2

(1) $f(x)=\begin{cases}\frac{1}{8} & \text{for} \quad x=0,3\\ \frac{3}{8} & \text{for} \quad x=1,2\end{cases}$

(2) (i)

x	2	4	6	8	10	Total
$f(x)$	$\frac{1}{36}$	$\frac{4}{36}$	$\frac{10}{36}$	$\frac{12}{36}$	$\frac{9}{36}$	1

(2) (ii) $F(x)=\begin{cases}0 & \text{for} \quad x<2\\ \frac{1}{36} & \text{for} \quad 2\le x<4\\ \frac{5}{36} & \text{for} \quad 4\le x<6\\ \frac{15}{36} & \text{for} \quad 6\le x<8\\ \frac{27}{36} & \text{for} \quad 8\le x<10\\ 1 & \text{for} \quad 10\le x<\infty\end{cases}$ (iii) $\frac{13}{18}$ (iv) $\frac{31}{36}$

(3) $f(x)=\begin{cases}\frac{1}{4} & \text{for} \quad x=1,3\\ \frac{1}{16} & \text{for} \quad x=0,4\\ \frac{3}{8} & \text{for} \quad x=2\end{cases}$, $F(x)=\begin{cases}0 & \text{for} \quad x<0\\ \frac{1}{16} & \text{for} \quad 0\le x<1\\ \frac{5}{16} & \text{for} \quad 1\le x<2\\ \frac{11}{16} & \text{for} \quad 2\le x<3\\ \frac{15}{16} & \text{for} \quad 3\le x<4\\ 1 & \text{for} \quad 4\le x<\infty\end{cases}$

(4) (i) 8 (ii) $F(x)=\begin{cases}0 & \text{for} \quad x<0\\ \frac{1}{8} & \text{for} \quad 0\le x<1\\ \frac{3}{8} & \text{for} \quad 1\le x<2\\ 1 & \text{for} \quad 2\le x<\infty\end{cases}$ (iii) $\frac{7}{8}$

(5) (i)

x	-1	0	1	2	3
$f(x)$	0.15	0.20	0.25	0.25	0.15

(ii) $P(X<1)=0.35$ (iii) $P(X\ge 2)=0.40$

(6) (i) $\frac{1}{6}$ (ii) $\frac{17}{36}$ (iii) $\frac{5}{6}$

(7) (a)

x	0	1	2	3	4
$f(x)$	$\frac{1}{2}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{10}$

(b) $\frac{4}{5}$ (c) $\frac{2}{5}$

EXERCISE 11.3

(1) 4

(2) (i)0.16 (ii) 0.3 (iii) 0.75

(3) (i) $\frac{1}{400}$ (ii) $F(x)=\begin{cases} 0 & \text{for} \quad x<200 \\ \frac{x}{400}-\frac{1}{2} & \text{for} \quad 200\le x\le 600 \\ 1 & \text{for} \quad x>600 \end{cases}$

(iii) $\frac{1}{2}$

(4) (i) $\frac{1}{3}$ (ii) $1-e^{-\frac{x}{3}}$ (iii) $1-e^{-1}$ (iv) $e^{-\frac{5}{3}}$ (v) $1-e^{-\frac{4}{3}}$

(5) (i) $F(x)=\begin{cases} 0 & x\le -1 \\ \frac{x^2}{2}+x+\frac{1}{2} & -1\le x<0 \\ -\frac{x^2}{2}+x+\frac{1}{2} & 0\le x<1 \\ 1 & 1\le x \end{cases}$ (ii) 0.75

(6) (i) $f(x)=F'(x)=\begin{cases} 0 & x<0 \\ \frac{1}{2}(2x+1) & 0\le x<1 \\ 0 & 1\le x \end{cases}$ (ii) 0.285

EXERCISE 11.4

(1) (i) $2.3, 2.81$ (ii) $1.67, 0.56$ (iii) $\frac{5}{3}, \frac{1}{18}$ (iv) $2, 4$

(2) $\frac{8}{7}$

x	0	1	2
$f(x)$	$\frac{1}{7}$	$\frac{4}{7}$	$\frac{2}{7}$

(3) $7, 16$

(4) $2, 1$

x	0	1	2	3	4
$f(x)$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

(5) 15 minutes (6) $\frac{1}{3}$ (7) $\frac{1}{2}, \frac{1}{8}$ (8) Loss ₹. 0.50

EXERCISE 11.5

(1) (i) $\dfrac{160}{729}$ (ii) $210\left(\dfrac{1}{5}\right)^4\left(\dfrac{4}{5}\right)^6$ (iii) $\begin{pmatrix}9\\7\end{pmatrix}\left(\dfrac{1}{2}\right)^7\left(\dfrac{1}{2}\right)^2$ (2) (i) $\begin{pmatrix}10\\4\end{pmatrix}\left(\dfrac{1}{4}\right)^4\left(\dfrac{3}{4}\right)^6$ (ii) $1-\dfrac{3^{10}}{4^{10}}$

(3) (i) $50,\ 25$ (ii) $40,\ \dfrac{100}{3}$ (4) $\dfrac{270}{1024}$ (5) (i) $1-0.95^{10}$ (ii) $\begin{pmatrix}10\\2\end{pmatrix}(0.05)^2(0.95)^8$

(6) (i) $\begin{pmatrix}12\\10\end{pmatrix}(0.9)^{10}(0.1)^2$ (ii) $2.1(0.9)^{11}$ (iii) $1-\left[2.1(0.9)^{11}\right]$

(7) (i) $\begin{pmatrix}18\\x\end{pmatrix}\left(\dfrac{1}{3}\right)^x\left(\dfrac{2}{3}\right)^{18-x}$ (ii) $\begin{pmatrix}18\\3\end{pmatrix}\left(\dfrac{1}{3}\right)^3\left(\dfrac{2}{3}\right)^{15}$ (iii) $1-\dfrac{20}{3}\left(\dfrac{2}{3}\right)^{17}$

(8) $\begin{pmatrix}6\\x\end{pmatrix}\left(\dfrac{1}{3}\right)^x\left(\dfrac{2}{3}\right)^{6-x},\ 2,\ \dfrac{2}{\sqrt{3}}$ (9) $1,\ \dfrac{4}{5}$

EXERCISE 11.6

1	2	3	4	5	6	7	8	9	10
(2)	(4)	(2)	(4)	(4)	(2)	(4)	(3)	(2)	(1)
11	12	13	14	15	16	17	18	19	20
(4)	(4)	(1)	(2)	(1)	(1)	(4)	(4)	(2)	(1)

Chapter 12

Exercise 12.1

1. (i) Yes, $*$ is binary on $\mathbb{R}$ (ii) Yes, $*$ is binary on A

   (iii) No, $*$ is not binary on $\mathbb{R}$

2. No, $*$ is not binary on $\mathbb{Z}$ 3. $\dfrac{-88}{15}$

4. Yes, usual multiplication is binary on A

5. (i) The given operation $*$ is closure and commutative but not associative on $\mathbb{Q}$.

   (ii) Identity does not exist and so inverse does not exist.

6.

$*$	a	b	c
a	b	c	a
b	c	b	a
c	a	a	c

7. No. The given operation is not commutative and associative

8. (i) $A\vee B=\begin{bmatrix}1&1&1&1\\1&1&1&1\\1&0&0&1\end{bmatrix}$ (ii) $A\wedge B=\begin{bmatrix}0&0&0&0\\0&0&0&0\\1&0&0&1\end{bmatrix}$

(iii) $(A \vee B) \wedge C = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ (iv) $(A \wedge B) \vee C = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

10. (i) It is commutative and associative.
    (ii) Identity and Inverse is exist.

Exercise 12.2

1. (i) $\neg p$: Jupiter is not a planet (ii) $p \wedge \neg q$: Jupiter is a planet and India is not an Island.
   (iii) $\neg p \vee q$: Jupiter is not a planet or India is an Island.
   (iv) $p \rightarrow \neg q$: If Jupiter is a planet then India is not an Island.
   (v) $p \leftrightarrow q$ Jupiter is a planet if and only if India is an Island.
2. (i) $\neg p \wedge q$ (ii) $p \vee \neg q$ (iii) $p \wedge q$ (iv) $\neg p$
3. (i) $p \rightarrow q$ is $\mathbb{T}$ (ii) $p \vee q$ is $\mathbb{F}$ (iii) $\neg p \vee q$ is $\mathbb{T}$ (iv) $p \wedge q$ is $\mathbb{F}$
4. (i), (iii) and (iv) are propositions
5. (i) **Converse:** If x and y are numbers such that $x^2 = y^2$ then $x = y$.
   Inverse: If x and y are numbers such that $x \neq y$ then $x^2 \neq y^2$.
   Contra positive: If x and y are numbers such that $x^2 \neq y^2$ then $x \neq y$.
   (ii) **Converse:** If a quadrilateral is a rectangle then it is a square.
   Inverse: If a quadrilateral is not a square then it is not a rectangle.
   Contrapositive: If a quadrilateral is not a rectangle then it is not a square.
6. (i) Truth table for $\neg p \wedge \neg q$

p	q	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

(ii) Truth table for $\neg(\neg p \wedge \neg q)$

p	q	$\neg q$	$p \wedge \neg q$	$\neg(\neg p \wedge \neg q)$
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

(iii) Truth table for $(p \vee q) \vee \neg q$

p	q	$\neg q$	$p \vee q$	$(p \vee q) \vee \neg q$
T	T	F	T	T
T	F	T	T	T
F	T	F	T	T
F	F	T	F	T

(iv) Truth table for $(\neg p \rightarrow r) \wedge (p \leftrightarrow q)$

p	q	r	$\neg p$	$(\neg p \rightarrow r)$	$p \leftrightarrow q$	$(\neg p \rightarrow r) \wedge (p \leftrightarrow q)$
T	T	T	F	T	T	T
T	T	F	F	T	T	T
T	F	T	F	T	F	F
T	F	F	F	T	F	F
F	T	T	T	T	F	F
F	T	F	T	F	F	F
F	F	T	T	T	T	T
F	F	F	T	F	T	F

7. (i) Contradiction (ii) Tautology (iii) Contingency (iv) Tautology

12. $p \rightarrow (q \rightarrow p)$ is a Tautology.

13. Yes. The statements are logically equivalent.

Exercise 12.3

Choose the appropriate answer from the given distractors.

Q	1	2	3	4	5	6	7	8	9	10
A	(2)	(3)	(2)	(4)	(2)	(2)	(3)	(4)	(3)	(2)
Q	11	12	13	14	15	16	17	18	19	20
A	(4)	(1)	(3)	(3)	(3)	(2)	(4)	(3)	(1)	(4)

GLOSSARY

CHAPTER 7
Application of Differential Calculus

related rates	சார்ந்த வீதங்கள்
mean value theorem	இடை மதிப்புத் தேற்றம்
indeterminate forms	தேறப்பெறாத வடிவங்கள்
stationary points	நிலைப் புள்ளிகள்
critical points	மாறுநிலைப் புள்ளிகள்
monotonicity of functions	ஓரியல்புச் சார்புகள்
absolute extremum	மீப்பெரு அறுதி
relative extremum	இடஞ்சார்ந்த அறுதி
Concave	குழிவு
Convex	குவிவு
point of inflection	வளைவு மாற்றுப் புள்ளி
Symmetry	சமச்சீர்த் தன்மை

CHAPTER 8
Differential and Partial Derivatives

Differential	வகையீடு
Partial Derivatives	பகுதி வகைக்கெழு
Harmonic	சீரான
Homogeneous	சமபடித்தான
Absolute error	தனிப்பிழை
Relative error	சார் பிழை
Percentage error	சதவீத பிழை

CHAPTER 9
Applications of Integration

Definite integral	வரையறுத்தத் தொகை
Reduction formula	குறைப்பு சூத்திரம்
Gamma integral	காமா தொகையிடல்
Bounded region	இடைப்பட்ட பகுதி

CHAPTER 10
Ordinary Differential Equations

order	வரிசை
Linear	நேரியல்
Degree	படி
arbitrary constant	ஏதேனுமொரு மாறிலி
dependent variable	சார்ந்த மாறி
independent variable	சாரா மாறி
integrating factor	தொகையீட்டுக் காரணி
homogeneous function	சமபடித்தான சார்பு

CHAPTER 11
Probability Distributions

bernoulli random variable	பெர்னோலி சமவாய்ப்பு மாறி
binomial distribution	ஈருறுப்பு பரவல்
binomial random variable	ஈருறுப்பு சமவாய்ப்பு மாறி
continuous random variable	தொடர்நிலை சமவாய்ப்பு மாறி
cumulative distribution function	குவிவு பரவல் சார்பு
discrete random variable	தனிநிலை சமவாய்ப்பு மாறி
mathematical expectation	கணித எதிர்பார்ப்பு
probability density function	நிகழ்தகவு அடர்த்திச் சார்பு
probability mass function	நிகழ்தகவு நிறைச்(செறிவு) சார்பு
random variable	சமவாய்ப்பு மாறி

CHAPTER 12
Discrete Mathematics

Absorption law	ஈர்ப்பு விதி
Algebraic structure	இயற்கணித அமைப்பு
Biconditional statement	இரு நிபந்தனைக் கூற்று
Binary Operation	ஈருறுப்பு செயலி
Boolean Algebra	பூலியன் இயற்கணிதம்
Boolean Matrix	பூலியன் அணி
Coding theory	குறியீட்டுக் கோட்பாடு
Compound statement	கூட்டுக் கூற்று
Conditional statement	நிபந்தனைக் கூற்று
Conjunction	இணையல்
Contradiction	முரண்பாடு
Contra positive	நேர்மாறு
Disjunction	பிரிப்பிணையல்
Duality	இருமை இயல்பு (அ) இரட்டைத் தன்மை
Hypothesis	கருதுகோள்
Involution law	உட்சுழற்சி விதி
Logical connectives	தர்க்க இணைப்புகள்
Logical equivalent	தர்க்க சமானமானவை
Negation	மறுப்பு
Paradox	முரண்படு மெய்மை
Simple statement	தனிக்கூற்று
Tautology	மெய்மம்
Truth table	மெய்மை அட்டவணை

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Text Book Development Team - Class 12

Mathematics - Volume II

Domain Experts

Dr. S. Udayabaskaran,
Professor, Department of Mathematics,
Vel Tech Rangarajan Dr. Sagunthala
R&D Institute of Science and Technology
(Deemed to be University),
Avadi, Chennai.

Dr. R. Murthy,
Principal (Retd.),
Govt. Arts College,
Uthiramerur,Kanchipuram.

Dr. E. Chandrasekaran,
Professor, Department of Mathematics,
Vel Tech Rangarajan Dr. Sagunthala
R&D Institute of Science and Technology
(Deemed to be University),
Avadi, Chennai.

Dr.Youvaraj,
Former Director,Dept. of Mathematics,
RIASM, University of Madras,Chennai.

Dr. T.N.Shanmugam,
Professor(Retd.),
Department of Mathematics,
Anna University, Chennai.

Dr.A.Rahim Basha,
Associate Professor (Retd.)
Department of Mathematics
Presidency College, Chennai

Dr. G. Palani,
Assistant Professor,
Department of Mathematics,
Dr.Ambedkar Govt Arts College,Vyasarpadi
Chennai.

Reviewers

Dr. V. Thangaraj,
Former Director of RIASM(Retd.),
University of Madras,
Chennai.

Dr. E.S. Jambulingam,
Principal (Retd.),
Government Arts College,
Tiruttani.

Dr. P.R. Vittal,
Professor(Retd.),
Former Principal & HOD,
Department of Mathematics,
RKM Vivekananda College Mylapore,
Chennai.

Academic Coordinator

B. Tamilselvi,
Deputy Director, SCERT,
Chennai.

Subject Coordinator

Nevetha Selvaraj,
Assistant Professor,
SCERT, Chennai.

Authors

M. Madhivanan,
Head Master,
Govt. Model Hr. Sec. School,
Karimangalam,
Dharmapuri.

N. Kalaiselvam,
PGT, Chennai Girls Hr. Sec. School,
Nungambakkam, Chennai.

A. Balamurugan,
PGT, Adhiyaman GBHSS,
Dharmapuri.

S. Panneerselvam,
PGT,(Retd.),
GHSS, GKM Colony,
Chennai.

S.P.Karthikeyan,
PGT, Gandhi Memorial Hr. Sec. School,
Thiruvennainallur, Villupuram.

C.S. Veeragavan,
PGT,
Sri Krishnaa Matric Hr. Sec. School,
TVS Nagar,Coimbatore-25.

ICT Coordinator

D. Vasu Raj,
PGT & HOD,
Dept of Mathematics,
KRM Public School,Chennai.

Qr Code Management Team

R. Jaganathan
SGT, PUMS - Ganesapuram, Polur,
Thiruvannamalai.

A. Devi Jesintha,
B.T. Asst, G.H.S, N.M. Kovil, Vellore.

V. Padmavathi, , B.T.
Asst.,Government High School,,
Vetriyur,Thirumanur, Ariyalur.

Illustration, Layout & Typist

S. Manoharan
V.V. Graphics,
Palavanthangal,
Chennai-114

Wrapper Design

Kathir Arumugam

In-House QC

Yogesh B.
C. Jerald Wilson
Kamatchi Balan Arumugam
Rajesh Thangappan
V. Sridhar
Arun Kamaraj Palanisamy

Coordination

Ramesh Munisamy

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